Numerical solutions of SPDEs with boundary noise

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Except where otherwise indicated, this thesis is my own original work.

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To my nan. R.I.P.

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Abstract

Galerkin finite element method is a technique for approximating solutions to stochastic partial differential equations (SPDEs) that has been extensively studied in the literature. In this thesis, we extend the scheme to solve the case where noise enters through the boundary of the domain. We prove that the optimal convergence rate is achieved for semi-linear parabolic SPDEs with random Neumann boundary conditions. Considering the advection-diffusion equation with boundary noise, we show that solutions are useful for simulating solute dynamics in arteries where the vessel walls are treated as flexible. We also investigate SPDEs with Dirichlet boundary noise. The solutions do not exist in a Sobolev space but in a weighted Sobolev space. For the one-dimensional heat equation with white noise, we show that a numerical scheme that combines Galerkin finite element method in space and discontinuous Galerkin stepping in time converges at an optimal rate.

Contents

List of Figures

List of Tables

Notation and symbol

a.e.	almost everywhere
a.s.	almost surely
w.r.t.	with respect to
$(\Omega, \mathcal{F}, \mathbf{P})$	Probability space
$\mathbf{P}, \mathbf{E}$	Probability measure and expectation
$\gamma(H, V)$	Space of all γ-radonifying operators: $H \to V$
C_c^∞	Space of infinite differentiable functions with compact support
∂D	Boundary of the Euclidean domain D
$\mathcal{D}(A)$	Domain of the operator A
$\mathbf{D}^\alpha$	Derivative operator with α order
H	Abstract Hilbert space, typically $H = L^2(D)$
H^r	Hilbert space with order r
H^r_ζ	Hilbert space with power-type weight, $H^r_\zeta = W^{k,2}_\zeta$
$\dot{H}^r$	$\mathcal{D}(A^{\frac{r}{2}}) \subset H$
H^*	Dual of space H
L^p	L^p space, typically $L^p(D)$ or $L^p(\partial D)$
$L(Z, V)$	Space of all linear, bounded operators: $Z \to V$
$L_2(H, V)$	Space of all Hilbert-Schmidt operators: $H \to V$
$L^p(\Omega; H)$	Space of p-fold integrable mappings: $\Omega \to H$

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}^n$	n dimensional Euclidean space
$W^{k,p}$	Sobolev space of order $k \in \mathbb{N}$, typically $W^{k,p}(D)$
$W^{k,p}_{\zeta}$	Weighted Sobolev space with power-type weight
$W_H(t)$	Cylindrical Wiener process taking values in H
$\partial W^{k,p}$	$W^{k,p}(\partial D)$
$\Vert \cdot \Vert$	Norm of H
$\Vert \cdot \Vert_r$	Norm of space $\dot{H}^r$
$\vert \cdot \vert_r$	Semi-norm of space H^r
$\Vert \cdot \Vert_V$	Norm of space V
$\vert \cdot \vert_V$	Semi-norm of space V
$(\cdot, \cdot)_V$	Inner product on space V
$(\cdot, \cdot)$	Inner product on $L^2(D)$
$(\cdot, \cdot)_1$	Inner product on $\dot{H}^1$
$\langle \cdot, \cdot \rangle$	Dual pairing

Chapter 1

Introduction

In the thesis we consider Galerkin approximation to the solutions of partial differential equations (PDEs) with boundary noise. An example of a heat equation with the boundary noise on a smooth convex domain $D \subset \mathbb{R}^n$ with boundary ∂D is given by

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,t) &= \Delta u(x,t) + f(x,t) + b(x,t)\dot{w}_1(x,t); \quad \text{on } (0,T] \times D,\\
\mathcal{B}u(x,t) &= c(x,t)\dot{w}_2(x,t); \quad \text{on } (0,T] \times \partial D,\\
u(0,x) &= u_0(x); \quad \text{on } D.
\end{aligned}$$

The differential operator $\mathcal{B}$ is a boundary condition defined as $\mathcal{B}u := -\frac{\partial u}{\partial \vec{n}}$ (Neumann) or $\mathcal{B}u := u$ (Dirichlet) where $\vec{n}$ is the normal vector on the surface ∂D. The processes $(\dot{w}_i(x,t))_{t\geq 0}$ for $i = 1, 2$ are space-time (white) noises that will be modelled by (Hilbert space valued) Wiener processes or cylindrical Wiener processes.

For PDEs with Neumann boundary noise, the results on the existence, uniqueness and regularity of solutions were obtained in a few papers under various conditions [12, 26, 13]. Some generalization on the nonlinear parabolic PDEs were considered in [6, 29, 43, 14] with certain constraints on the solution spaces and the boundary dynamics. Moreover, optimal control problems were considered under the framework of PDEs with the Neumann boundary noise in [17, 55], and PDEs with the Lévy boundary noise were considered in [34].

However, for PDEs with Dirichlet boundary noise, it was shown that, even in the simplest case of the one-dimensional domain $D = [0, 1]$, that one cannot obtain $L^2(0, 1)$-valued solutions to the classic heat equation. Hence, a bigger space, i.e., a weighted Sobolev space, has to be considered as where the solution would exist [13, 44, 1]. In [40], a thorough review of the relevant results on the existence and uniqueness of solutions in one dimension is given. The regularity of the solution in higher dimension was considered in [39] and it was shown that the solution would "explode" inevitably.

1.1 Numerical solutions to SPDEs

Several numerical schemes have been attempted in approximating the solutions to SPDEs with the homogeneous boundary conditions. Among them, Galerkin finite element method (FEM) and collocation method were the most popular and their variations were largely considered in the literature. Galerkin FEM for the linear SPDEs was discussed in [50] where an optimal convergence rate was obtained. The same numerical scheme was applied to semilinear SPDEs in [30]. The solutions in [30] were more regular (of order $1 + r, r \in [0, 1)$) with some regularity constraints on the terms in the SPDEs and the initial conditions. The error estimates of the approximation were optimal in this setup.

The collocation method was considered in [3, 11] and [20]. Technically the collocation method can be seen as a spectral FEM and the papers were concerned with the discretization of the random variable space. In [53], Wiener chaos expansion and stochastic collocation methods were presented with the applications on solving SPDEs. The literature mentioned above focused on seeking a "best" representation of the random fields or stochastic processes or seeking a numerical scheme with the "best" convergence rate, which is not the focus of this thesis. We are not concerned with any particular technique but how the boundary noises would affect the approximation with a classic Galerkin FEM approach.

Besides the two methods extended from PDEs, other numerical schemes with empirical results are proposed too. For example, one is the probabilistic approach such as Monte Carlo in simulating the moments of the solutions. The other is based on series expansion aiming to recover the density of the solutions. These approaches were discussed and compared in [54].

All the literature mentioned above dealt with the initial value problem, i.e., the equations driven by spatial or time noises. Except for a most recent paper [22], there is no known literature in the numerical solutions to the boundary value problem for the SPDEs. However, in this paper, a simulation only approach was used to demonstrate the convergence rate. There are potential issues with the empirical approach since it lacks theoretical convergence proofs and, as such, is restricted to only a few examples. Moreover, such an approach cannot identify the conditions that constrain or limit the convergence rates and the ways in which this occurs.

We are intrigued by the questions of how the errors of numerical approximations would be subject to the irregularity introduced by the boundary noises. Hence, in the thesis, we consider not only the equations with Neumann boundary noises but also those with Dirichlet boundary noises. No results have appeared on these questions in the literature to date. We contribute in proving the optimal convergence rate of Galerkin FEM in approximating the solutions of semi-linear SPDEs with Neumann boundary noise. It can be seen as an extension of [30] where the semi-linear SPDEs with the homogeneous boundary conditions are considered. We also show that the framework can be applied to a useful application in modeling blood solute dynamics. It extends the work of [37] by offering a solution in a more realistic setup. Finally we prove the optimal convergence rate of Galerkin time stepping scheme in solving a one dimensional heat equation with Dirichlet boundary noise. It is a new case that has not been considered before to our best knowledge.

1.2 Thesis Outline

We start with a brief review of the background material in **Chapter 2**. It covers the necessary definitions and results. In **Chapter 3**, we formulate a semi-linear PDE with Neumann boundary noise as a stochastic evolution equation. We also recall some known results on the existence, uniqueness and regularity of its solution. Moreover, we pose the conditions required on our solutions for the numerical approximations to work.

In **Chapter 4**, we present one of main results of the thesis. We prove that a numerical scheme using backward time stepping combined with the Galerkin finite element method (FEM) achieves the optimal convergence rate in approximating the solutions to the semi-linear PDEs with Neumann boundary noise. Moreover, we demonstrate that our theoretical convergence results are correct through some numerical experiments. In **Chapter 5**, we show that our general results can be of great use in modeling solute dynamics in blood vessels through the application to an advection-diffusion equation with boundary noise. In **Chapter 6**, we explore numerical approximations the difficult case of PDEs with Dirichlet boundary noise. While the case is limited to the one dimensional heat equation, it gives much insight in the assumptions required to achieve convergence of approximations to the solution. As an interesting side result, we also show the regularity of the solution is bounded by the "weight" of the solution space. We show that with Galerkin time stepping, the optimal convergence rate is achieved both spatially and temporally. The numerical experiments are performed to verify our results as well.

In the last chapter, we identify the assumptions and limitations of the numerical approximation of the solutions to the PDEs with boundary noises. We also propose some future ideas on how the results of this thesis could be extended.

1.3 Results overview

In the first instance, we consider Galerkin approximations to the solutions of a stochastic partial differential equation (SPDE) in a smooth convex domain $D \subseteq \mathbb{R}^d$ with boundary noise given by

$$\begin{aligned} dU(t,x) + &[\mathcal{A}U(t,x) + f(t,x,U(t,x))]\,dt \\ &= \bar{b}(t,x,U(t,x))dw_1(t,x) \quad \text{in } [0,T] \times D, \\ \mathcal{B}U(t,x) &= \bar{c}(t,U(t,x))dw_2(t,x) \quad \text{in } [0,T] \times \partial D, \\ U(0,x) &= U_0(x), \text{ in } D, \end{aligned}$$

where $\mathcal{A}$ is a generalized differential operator with the divergence form given by

$$\mathcal{A} = \nabla \cdot (\mathbf{a}\nabla) + a_0(t,x), \tag{1.1}$$

with $\mathbf{a} = (a_{ij})_{i,j=1,\ldots,d}$ being a uniformly positive definite and symmetric matrix of $d \times d$, i.e., there exists a constant θ such that

$$\sum_{i,j=1}^{d} a_{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2, \quad \text{for all } x \in \overline{D},\ t \in [0,T],\ \xi \in \mathbb{R}^d.$$

The operator $\mathcal{A}$ is neither compact nor having a compact inverse. The conormal operator $\mathcal{B}$ is given by

$$\mathcal{B} = \mathbf{n} \cdot (\mathbf{a}\nabla).$$

With its solution existing in Hilbert space of order $r \in [0, 1)$, the error between the numerical approximation with spatial semi-discretization and the "true" solution is bounded by Ch^r, where $C > 0$ does not depend on the mesh size $h \in (0, 1]$ or r. Together with the backward time stepping, the full discretization has the convergence rate of $C(h^r + k^{\frac{r}{2}})$, where $k \in (0, 1]$ is the time stepping size and the constant $C > 0$ does not depend on h, k or r. The convergence rates are called *optimal*. The results are presented in **Theorem 4.6** and **Theorem 4.9**.

We propose that this type of SPDEs could be used in modeling solute dynamics in blood vessels by formulating the dynamics as an advection-diffusion equation with Neumann boundary noises. Given the velocity of blood flow $\mathbf{u}$, for the *wall-free* model, the equation system is given by

$$\begin{aligned}
&\frac{\partial C_f}{\partial t} - \nabla \cdot (\boldsymbol{\mu}_f \nabla C_f) + \mathbf{u}\nabla C_f = f_f, \quad \mathbf{x} \in \Omega_f,\ t \in (0, T],\\
&\mathbf{n} \cdot (\boldsymbol{\mu}_f \nabla C_f) + \zeta C_f = \zeta \kappa_w, \quad \mathbf{x} \in \Gamma_w,\ t \in (0, T],\\
&C_f = 0 \quad \text{on } \partial\Omega_f \backslash \Gamma_w,\ t \in (0, T],\\
&C_f = C_{f,0}, \quad \mathbf{x} \in \Omega_f,\ t = 0,
\end{aligned}$$

with the boundary condition

$$\mathbf{n} \cdot (\boldsymbol{\mu}_f \nabla C_f) = \tilde{\zeta} + \dot{W}(t), \ \ \mathbf{x} \in \Gamma_w, t \in (0, T].$$

For the *fluid-wall* model, the equation system proposed is

$$\begin{aligned}
&\frac{\partial C_f}{\partial t} - \nabla \cdot (\boldsymbol{\mu}_f \nabla C_f) + \mathbf{u} \cdot \nabla C_f = f_f \quad \text{in } \Omega_f,\ t \in (0, T],\\
&C_f = 0 \quad \text{on } \partial\Omega_f \backslash \Gamma,\ t \in (0, T],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial C_w}{\partial t} - \nabla \cdot (\mu_w \nabla C_w) = f_w \quad \text{in } \Omega_w,\ t \in (0, T],\\
&C_w = 0 \quad \text{on } \partial\Omega_w \backslash \Gamma,\ t \in (0, T],
\end{aligned}$$

with the boundary condition

$$\begin{aligned} \mathbf{n}_w \cdot (\mu_w \nabla C_w) &= \tilde{\zeta} + \dot{W}(t) \quad \text{on } \Gamma, \\ \mathbf{n}_f \cdot (\boldsymbol{\mu}_f \nabla C_f) &= \tilde{\zeta} + \dot{W}(t) \quad \text{on } \Gamma. \end{aligned}$$

The solute concentrate C_f and C_w achieve an equilibrium state perturbed by the noises as shown in **Figure 5.4** and **Figure 5.5**.

The last part of thesis is concerned with the numerical solutions to the one-dimensional heat equation with Dirichlet noise given by

$$\begin{aligned} \frac{\partial U}{\partial t}(t,x) &= \Delta U(t,x) \qquad \text{on } [0,T] \times \mathbb{R}_+, \\ U(t,0) &= \dot{W}_t \qquad \text{on } [0,T], \\ U(0,x) &= u_0 \qquad \text{on } \mathbb{R}_+. \end{aligned}$$

We show that the maximum regularity $r \in [0,1)$ that a solution could achieve in a weighted Sobolev space is bounded by the weight in **Theorem 6.4**. The convergence rate is optimal for the spatial semi-discretization, i.e., bounded by Ch^r, where $C > 0$ does not depend on $h \in (0,1]$ or $r \in [0,1)$. With the solution not only weak in space but also weak in time, we show that the numerical scheme together with Galerkin time stepping has the optimal convergence rate $C(h^r + k^{\frac{r}{2}})$, where $C > 0$ does not depend on $h \in (0,1]$, $k \in (0,1]$ or $r \in [0,1)$. These results are presented in **Theorem 6.5** and **Theorem 6.6**.

Apart from the theoretical error estimates, we perform the numerical experiments on the one-dimensional heat equations with Neumann boundary noise and Dirichlet boundary noise respectively. Hence the theoretical results obtained are numerically verified as shown in **Figure 4.1- 4.4** and **Figure 6.1- 6.4**.

Chapter 2

Useful background results

In this chapter we briefly give some standard background definitions, results, and assumptions that will hold or be used in the rest of thesis. References to works containing more details are given throughout.

2.1 Sobolev-Slobodeckij space

The *Sobolev-Slobodeckij space* is a Sobolev space of fractional order, also called *fractional Sobolev space* (e.g., see [18]).

Definition 2.1. Let $s \in (0,1)$ and $p \in [1,\infty)$, for $D \subset \mathbb{R}^n$, *Sobolev-Slobodeckij spaces (fractional Sobolev spaces)* $W^{s,p}(D)$ are defined as

$$W^{s,p}(D) := \left\{ f \in L^p(D) : \frac{|f(x) - f(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(D \times D) \right\}.$$

In other words, an intermediate space between $L^p(D)$ and $W^{1,p}(D)$, equipped with the norm

$$\|f\|_{W^{s,p}(D)} := \left(\int_D |f|^p dx + \int_D \int_D \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dxdy \right)^{\frac{1}{p}}$$

where the second term in the parenthesis

$$|f|_{W^{s,p}(D)} := \left(\int_D \int_D \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dxdy \right)^{\frac{1}{p}}$$

is the *semi-norm* and when $p = 2$ we write $|f|_s$.

Sobolev-Slobodeckij space can be extended to the case where $s \in \mathbb{R}^+$. Let $C_c^\infty(D)$ denote the space of infinite differentiable functions with the compact support on D. The *generalized derivative* $\phi = \mathbf{D}^\alpha f$ of a function f is defined as

$$\int f(x)\mathbf{D}^\alpha g(x)dx = (-1)^{|\alpha|}\int \phi(x)g(x)dx, \quad \forall g \in C_c^\infty(D)$$

where α is a multiindex such that

$$\alpha = \{\alpha_n\}, n = 1, 2, \dots, N, \alpha_n \in \mathbb{N} \cup \{0\},$$
$$|\alpha| = \sum_{n=1}^{N} \alpha_n,$$

$x = (x_1, \dots, x_N)$ and the differential operator denoted by

$$\mathbf{D}^\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_N}}{\partial x_{\alpha_2}\cdots\partial x_{\alpha_N}}.$$

When $s > 1$ and is not an integer, we write $s = m + \sigma$, where $m \in \mathbb{N}$ and $0 < \sigma < 1$. In this case, the *Sobolev-Slobodeckij* space is defined:

$$W^{s,p} := \{f \in W^{m,p}(D) : \mathbf{D}^\alpha f \in W^{\sigma,p}(D)\ \forall \alpha\ s.t.\ |\alpha| = m\}$$

We recall the following properties of Sobolev-Slobodeckij spaces (see [41]).

Proposition 2.2. *Let $s \in \mathbb{R}^+$, $p \in [1, \infty)$. Then*

1. *$W^{s,p}$ is separable;*
2. *If $p > 1$, $W^{s,p}$ is reflexive;*
3. *$W^{s,2}$ is a Hilbert space denoted by H^s endowed with the norm $\|\cdot\|_{H^s}$.*

2.2 The Laplacian and associated semigroup

The Hilbert spaces $H^s := H^s(D) = W^{s,2}(D)$ where $D \subset \mathbb{R}^n$ and $S \in \mathbb{R}$ are related to the fractional Laplacian operator $(-\Delta)^{\frac{s}{2}}$, where Δ is a second order differential operator defined by $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Proposition 2.3. *[18, Proposition 3.6] Let $s \in (0, 1)$ and $f \in H^s$. Then*

$$|f|_s^2 = C_{n,s}\|(-\Delta)^{\frac{s}{2}} f\|^2.$$

where $C_{n,s}$ is a constant dependent on n and s.

More generally, consider an elliptic differential operator

$$\mathcal{A} = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c_0, \tag{2.1}$$

where a_{ij}, b_i and c_0 are C^∞ functions defined on the closure of the domain $\overline{D}$. Note that $\mathcal{A}$ is elliptic when the matrix $\mathbf{a} = (a_{ij})_{i,j=1,\ldots,n}$ is positive-definite and symmetric. A densely defined, self-adjoint and positive-definite operator A on a separable Hilbert space H with compact inverse is given by $\mathcal{A}$ such that $Au + f(u) = \mathcal{A}u$ for $u \in \mathcal{D}(A)$ where $\mathcal{D}(A)$ denotes the domain of A. It is shown in Chapter 2 [30] that the differential operator of form (2.1) might not be self-adjoint and positive definite. However, it can be split in two parts as

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) u,$$
$$f(u) = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} u + c_0 u,$$

such that A satisfies the conditions above.

Hence we can define the fractional operator $A^{\frac{s}{2}}$ with $s > 0$ as a mapping: $\mathcal{D}(A^{\frac{s}{2}}) \subset H \to H$, where H is a separable Hilbert space. The domain of $A^{\frac{s}{2}}$ with $s > 0$ is denoted by $\dot{H}^s := \mathcal{D}(A^{\frac{s}{2}})$. By the spectral theorem on the self-adjoint and linear compact operator $-A$ on Hilbert space H, there exists a sequence of increasing eigenvalues λ_n and an orthonormal basis of eigenvectors e_n in H (see [5]) such that

$$Ae_n = \lambda_n e_n, \quad n \in \mathbb{N}. \tag{2.2}$$

Hence the space $\dot{H}^s$ is a separable Hilbert space endowed with the norm

$$\|f\|_s^2 := \sum_{n=1}^{\infty} \lambda_n^s (f, e_n)^2 < \infty,$$

and the inner product

$$(\cdot,\cdot)_s := (A^{\frac{s}{2}}\cdot, A^{\frac{s}{2}}\cdot).$$

In order to have a characterization of the dual space $(\dot{H}^s)^*$, we introduce a set

$$\dot{H}^{-s} := \left\{ f = \sum_{n=1}^{\infty} f_n e_n : f_n \in \mathbb{R}, n \in \mathbb{N}, \text{such that } \|f\|_{-s}^2 := \sum_{n=1}^{\infty} \lambda_n^{-s} f_n^2 < \infty \right\},$$

and we define the operator A with the negative fractional power as

$$A^{-\frac{s}{2}} f = \sum_{n=1}^{\infty} \lambda_n^{-\frac{s}{2}} f_n e_n$$

for all $f = \sum_{n=1}^{\infty} f_n e_n \in \dot{H}^{-s}$. It holds that $H \subset \dot{H}^{-s}$ for $s > 0$. Hence $\dot{H}^{-s}$ is the largest set such that $A^{-\frac{s}{2}}$ is a mapping: $\dot{H}^{-s} \to H$. In this sense the domain of $A^{-\frac{s}{2}}$ is $\dot{H}^{-s}$. Hence similar to $\dot{H}^{s}$ above, $\dot{H}^{-s}$ is endowed with the inner product $(\cdot,\cdot)_{-s} := (A^{-\frac{s}{2}}\cdot, A^{-\frac{s}{2}}\cdot)$ and the norm $\|\cdot\|_s = \|A^{-\frac{s}{2}} \cdot\|$ where $\|\cdot\|$ denotes a L^2-norm.

Thus we have the next theorem which shows the characterization of the dual space $(\dot{H}^s)^*$.

Theorem 2.4. *[30, Theorem B.8] For $s > 0$, the dual space $(\dot{H}^s)^*$ is isometrically isomorphic to $\dot{H}^{-s}$. Moreover $\dot{H}^{-s}$ is a separable Hilbert space.*

Recall that A is a densely defined, self-adjoint and positive-definite operator with compact inverse on a separable Hilbert space H. Under these conditions, Hille-Yosida theorem implies that the operator $-A$ is the infinitesimal generator of a C_0-semigroup $(E(t))_{t\geq 0}$ on H. Moreover, the following properties hold for $(E(t))_{t\geq 0}$ and A^s.

Lemma 2.5. *[30, Lemma B.9] Let $(E(t))_{t\geq 0}$ be C_0-semigroup on H and its infinitesimal generator be $-A$ defined above, then the following properties hold true:*

(i) For any $\rho > 0$, it holds that

$$A^{\rho} E(t) x = E(t) A^{\rho} x, \qquad \forall x \in \dot{H}^{2\rho},$$

and there exists a constant $C = C(\rho)$ such that

$$\|A^{\rho} E(t)\| \leq C t^{-\rho}, \qquad \forall t > 0$$

(ii) For any $0 \leq \rho \leq 1$ there exists a constant $C = C(\rho)$ such that

$$\|A^{-\rho} \left(E(t) - Id \right)\| \leq C t^{\rho}, \qquad \forall t > 0$$

(iii) For any $0 \leq \rho \leq 1$, there exists a constant $C = C(\rho)$ such that

$$\int_{t_1}^{t_2} \|A^{\frac{\rho}{2}} E(t_2 - s)x\|^2 ds \leq C(t_2 - t_1)^{1-\rho} \|x\|^2$$

2.3 Trace operators

When dealing with boundary value problems, it is important to prescribe the boundary values of a given function. Let $C^\infty(\overline{D})$ denote the set of infinite differentiable functions in the domain D, whose derivatives are bounded and uniformly continuous in D. For $u \in C^\infty(\overline{D})$, it has sense to consider the restriction $u|_{\partial D}$ to a lower dimensional manifold $\partial D \subset \mathbb{R}^{n-1}$. However, for $u \in W^{s,p}(D)$, the restriction does not have a general sense since the functions are only defined on D a.e. and the Lebesgue measure vanishes on ∂D. Hence we introduce the *trace* to formalise the restriction of functions in Sobolev space defined on $D \subset \mathbb{R}^n$ to the boundary ∂D of D. We now recall results of the *trace* and *normal trace* that can be found in [18] and [41]. These results hold under the Lipschitz assumption of domain D.

Definition 2.6. We say that a bounded open subset $D \subset \mathbb{R}^n$ is Lipschitz of class C^k if its boundary ∂D can be covered by a finite number of open hypercubes $B_j, j = 1, ..., m$, with an attached system of orthonormal Cartesian coordinates $(x^j) = (x_1^j, x_2^j, \ldots, x_n^j)$, in such a way that $B_j = \{x \in \mathbb{R}^n; |x_i^j| < a_j, i = 1, \ldots, n, a_j > 0\}$, and there exists Lipschitz function of class C^k $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $D \cap B_j = \{y \in B_j; x_j^n < \varphi_j((x_j)')\}$, where $(x^j)' = (x_1^j, x_2^j, \ldots, x_{n-1}^j) \in \mathbb{R}^{n-1}$.

Proposition 2.7 (*Traces*)**.** *Let $p \in [1, \infty)$, $s > 1/p$ and $D \subset \mathbb{R}^n$ of class C^0. There exists a unique linear and continuous* trace operator $\tau_0 : W^{s,p}(D) \to L^p(\partial D)$ *such that $\tau_0 u = u$ on ∂D for any $u \in C^\infty(\overline{D})$, and there exists a constant $C > 0$ such that*

$$\|\tau_0 u\|_{L^p(\partial D)} \leq C \|u\|_{W^{s,p}(D)}$$

for all $u \in W^{s,p}(D)$.

Proposition 2.8 (*Normal traces*)**.** *Let $p \in [1, \infty)$, $s > 1 + 1/p$, $D \subset \mathbb{R}^n$ of class C^1, and $\vec{n}$ be the outward-oriented unit normal vector field on ∂D. There exists a unique* *linear and continuous* normal trace operator $\tau_1 : W^{s,p}(D) \to L^p(\partial D)$ *such that* $\tau_1 u = \frac{\partial u}{\partial \vec{n}}$ *on* ∂D *for any* $u \in C^\infty(\overline{D})$*, and there exists a constant* $C > 0$ *such that*

$$\|\tau_1 u\|_{L^p(\partial D)} \leq C\|u\|_{W^{s,p}(D)}$$

for all $u \in W^{s,p}(D)$ *.*

We note that $\tau_1 u = \tau_0(\vec{n} \cdot \nabla u)$. If ∂D is of class C^∞, the operator τ_0 is a mapping $W^{s,p}(D) \to W^{s-\frac{1}{p},p}(\partial D)$ and τ_1 is a mapping $W^{s,p}(D) \to W^{s-1-\frac{1}{p},p}(\partial D)$.

For Hilbert spaces $H^s, s \in \mathbb{R}$, the results above can be generalized to the domain $\mathbb{R}^n$.

Proposition 2.9. *Let* $s > \frac{1}{2}$*; then, any function* $u \in H^s(\mathbb{R}^n)$ *has a trace* $\tau_0 u$ *on the hyperplane* $\{x_n = 0\}$*, such that* $\tau_0 u \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$*. Moreover the trace operator* τ_0 *is surjective from* $H^s(\mathbb{R}^n)$ *to* $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$*.*

2.4 Weighted Sobolev space

The weighted Sobolev space is introduced to deal with the situation where the trace of a function does not exist due to conditions such as singular points on the boundary, unbounded domain and domains with non-smooth geometric shapes [31]. We define the weighted Sobolev space with the minimum assumption of the domain D as in [31]. The domain $D \subset \mathbb{R}^n$ is bounded with the boundary ∂D being a manifold of dimension $n-1$.

Definition 2.10. The *weighted Sobolev space* $W^{k,p}(D;\rho)$ with $k \in \mathbb{N} \cup \{0\}, p \in [1,\infty)$ is defined as the set of all functions $u(x)$ defined a.e. on $D \subset \mathbb{R}^n$, whose generalized derivatives $\mathbf{D}^\alpha u$ for order $\alpha \leq k$ satisfy

$$\int_D |\mathbf{D}^\alpha u(x)|^p \rho_\alpha(x) dx < \infty,$$

where $\rho_\alpha(x)$ are non-negative measurable functions on D.

If the weighted Sobolev space is equipped with the norm

$$\|u\|_{W^{k,p}(D;\rho)} = \left(\sum_{|\alpha| \leq k} \int_D |\mathbf{D}^\alpha u(x)|^p \rho_\alpha(x) dx \right)^{\frac{1}{p}},$$

it is a normed linear space.

We define the *weight* ρ as a vector with ρ_α being elements, given by

$$\rho = \{\rho_\alpha = \rho_\alpha(x), x \in D, |\alpha| < k\}.$$

Throughout the thesis, we assume that for every α, $\rho_\alpha(x)$ is fixed, i.e.,

$$\rho_\alpha(x) = \rho(x), \quad \text{for every } \alpha, \ |\alpha| < k.$$

The weight functions can have different relations with the domain D, either vanishing somewhere within the closure $\overline{D}$ or increasing to infinity. We consider a special class of weight given by $\rho(x) = (\epsilon(x))^\zeta$, where $c\delta(x) \leq \epsilon(x) \leq C\delta(x)$ for some $c, C > 0$, with $\delta(x)$ being the distance from the point $x \in D$ to the subset $M \subset \partial D$, i.e.,

$$\delta(x) := \inf_{y \in M} |x - y|.$$

The weight ρ is called a *power-type weight* and we denote $W^{k,p}_\zeta := W^{k,p}(D; \epsilon^\zeta)$. Thus, the space $W^{k,p}_\zeta$ is a weighted Sobolev space given by

$$W^{k,p}_\zeta = \left\{ u = u(x) : \int\limits_D |\mathbf{D}^\alpha u(x)|^p \epsilon^\zeta(x) dx < \infty, \quad \text{for all } |\alpha| \leq k \right\}.$$

It is equipped with the norm

$$\|u\|_{W^{k,p}_\zeta} = \left(\sum_{|\alpha| \leq k} \int\limits_D |\mathbf{D}^\alpha u(x)|^p \epsilon^\zeta(x) dx \right)^{\frac{1}{p}}. \tag{2.3}$$

Let $W^{k,p}_{\zeta,0}(D)$ denote the closure of $C^\infty_c(D)$ in the norm (2.3). We recall the following result with respect to the weighted Sobolev space $W^{k,p}_{\zeta,0}$.

Theorem 2.11. *[31, Theorem 3.9] The weighted Sobolev space $W^{k,p}_{\zeta,0}$ is a separable Banach space.*

If $p = 2$, the weighted Sobolev space $W^{k,p}_\zeta$ is a Hilbert space denoted by

H^k_ζ equipped with the inner product

$$(u,v)_{k,\zeta} = \sum_{|\alpha|\le k} \int_D \mathbf{D}^\alpha u(x)\epsilon^{\frac{\zeta}{2}}(x)\, \mathbf{D}^\alpha v(x)\epsilon^{\frac{\zeta}{2}}(x)dx.$$

The weighted fractional Sobolev space $W^{s,p}_\zeta, 0 < s < 1$ can be constructed as the intermediate between two weighted Sobolev spaces L^p_ζ and $W^{1,p}_\zeta$. Most results on the weighted Sobolev space $W^{k,p}_\zeta$ can be extended to the weighted fractional Sobolev space $W^{s,p}_\zeta$ in one dimensional domain. For the details, we refer readers to [32].

2.5 Cylindrical Wiener process and γ-radonifying operators

Cylindrical Wiener process and its properties are well-known in the literature of SPDEs, e.g., [25], [49], and [14]. We give a concise review in this section following [38] closely so that we can see how the *γ-radonifying operators*, *Hilbert-Schmidt operators* and the *covariance operators* are related.

We denote a probability space by $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$. Let V denote a separable Banach space with dual V^* and $L^p(\Omega; V)$ denote the space of the random variables $U : \Omega \to V$ with the norm $\|U\|_{L^p(\Omega;V)} = \left(\mathbf{E}\|X\|^p_V\right)^{\frac{1}{p}}$. If $V = \mathbb{R}$, we denote $L^p(\Omega) := L^p(\Omega; \mathbb{R})$.

Let H be a separable Hilbert space and $L(H,V)$ denote the space of all linear bounded operators from H to V, and its norm is given by

$$\|R\|_{L(H,V)} = \sup_{h\in H, \|h\|_H = 1} \{\|Rh\|_V\}.$$

We now define *γ-radonifying operators*:

Definition 2.12. Let γ be the standard Gaussian cylindrical measure on a separable Hilbert space H with an orthonormal basis $(h_n)_{n\in\mathbb{N}}$. A linear bounded operator $\Gamma \in L(H,V)$ is called *γ-radonifying* if the cylindrical measure $\gamma \circ \Gamma^{-1}$ extends to a Gaussian measure on the Banach space V. The space of all γ-

radonifying operators is denoted by $\gamma(H, V)$ and equipped with norm

$$\|\Gamma\|_{\gamma(H,V)} = \left(\mathbf{E} \left\| \sum_{k=1}^{\infty} \xi_k \Gamma h_k \right\|^2 \right)^{\frac{1}{2}},$$

where $(\xi_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of independent standard real normal random variables.

We refer the reader to [38] on how to define a Gaussian law on a Banach space. We have the following characterisation with regard to γ-radonifying operators:

Theorem 2.13. *[38, Theorem 5.2] Let γ be the standard Gaussian cylindrical measure on a separable Hilbert space H with orthonormal basis $(h_n)_{n\in\mathbb{N}}$ and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent standard real normal random variables. For $\Gamma \in L(H, V)$ the following are equivalent:*

1. *Γ is γ-radonifying;*
2. *the operator $Q := \Gamma\Gamma^* : V^* \to V$ is the covariance operator of a Gaussian measure μ on V;*
3. *the series $\sum_{k=1}^{\infty} \xi_k \Gamma h_k$ converges a.s. in V;*
4. *the series $\sum_{k=1}^{\infty} \xi_k \Gamma h_k$ in $L^p(\Omega; V)$ for some $p \in [1, \infty)$;*
5. *the series $\sum_{k=1}^{\infty} \xi_k \Gamma h_k$ in $L^p(\Omega; V)$ for all $p \in [1, \infty)$.*

Also we have for every $p \in [1, \infty)$

$$\int_V \|v\|^p \mu(dv) = \mathbf{E} \left\| \sum_{k=1}^{\infty} \xi_k \Gamma h_k \right\|^p \quad \textit{for } v \in V.$$

Moreover, the γ-radonifying operators have the *ideal property* as is shown by the following theorem.

Proposition 2.14. *[48, Proposition 2.3] Let $\tilde{V}$ be a real Banach space and $\tilde{H}$ be a real separable Hilbert space. If $B_1 \in L(\tilde{H}, H)$, $\Gamma \in \gamma(H, V)$ and $B_2 \in L(V, \tilde{V})$, then $B_2 \circ \Gamma \circ B_1 \in \gamma(\tilde{H}, \tilde{V})$ and $\|B_2 \circ \Gamma \circ B_1\|_{\gamma(\tilde{H},\tilde{V})} \leq \|B_2\|_{L(\tilde{H},H)} \|\Gamma\|_{\gamma(H,V)} \|B_1\|_{L(V,\tilde{V})}$.*

This property is useful and important in formulating (3.1) in the equation of homogeneous boundary conditions and the error estimates later on.

The next corollary shows that if V is a separable Hilbert space, then γ-radonifying coincides with *Hilbert-Schmidt operators*.

Corollary 2.15. *[38, Corollary 5.3] If H and V are separable Hiblert spaces, then the following are equivalent for $\Gamma \in L(H, V)$:*

1. *Γ is γ-radonifying;*

2. *Γ is Hilbert-Schmidt.*

We denote the space of all *Hilbert-Schmidt operators* from H to V by $L_2(H, V)$ and it is equipped with the norm $\|\Gamma\|_{L_2(H,V)} = \left(\sum_{k=1}^{\infty} \|\Gamma h_k\|^2\right)^{\frac{1}{2}}$ for an arbitrary orthonormal basis $(h_n)_{n\in\mathbb{N}}$ of H. We note that the space $L_2(H, V)$ is a separable Hilbert space with inner product denoted by $(\cdot, \cdot)_{L_2}$ such that $(\Gamma, T)_{L_2} = \sum_{k=1}^{\infty}(\Gamma h_k, T h_k)_V$. The equivalence in Corollary 2.15 can be easily seen by $\|\Gamma\|^2_{\gamma(H,V)} = \|\Gamma\|^2_{L_2(H,V)}$.

Moreover, the symmetric, positive definite operator Q in Theorem 2.13 is the covariance operator of a Gaussian measure $\mu = \gamma \circ \Gamma^{-1}$ on the Hilbert space V if and only if Γ is Hilbert-Schmidt.

Now we define the *cylindrical Wiener process* as follows:

Definition 2.16. Let H be a separable Hilbert space with inner product $(\cdot, \cdot)_H$. A *H-cylindrical Wiener process* is a family $(W_H(t))_{t\geq 0}$ of linear bounded operators from H to $L^2(\Omega)$ with the following properties:

1. $(W_H(t)h)$ is a scalar Brownian motion and $\mathcal{F}_t$ measurable for all $h \in H$ and $t \geq 0$. $(W_H(t))_{t\geq 0}$ is also called *adapted* to a given filtration $\{\mathcal{F}_t\}_{t\geq 0}$;

2. $\mathbf{E}(W_H(t_1)h_1 W_H(t_2)h_2) = (t_1 \wedge t_2)(h_1, h_2)_H$ for all $h_1, h_2 \in H$ and $t_1, t_2 \geq 0$.

We have a direct generalization of a real-valued Wiener process to the Banach space V.

Definition 2.17. An adapted V-valued stochastic process $(W(t))_{t\geq 0}$ is called a *Wiener process* if

1. $W(0) = 0$ $\mathbf{P}$-a.s.;

2. W had independent and stationary increments;

3. there exists a Gaussian covariance operator $Q : V^* \to V$ such that

$$W(t) - W(s) \stackrel{d}{=} \mathbf{N}(0, (t-s)Q) \quad \text{for all } 0 \le s \le t.$$

By Theorem 2.13, the covariance operator Q of V-valued Wiener process $(W(t))_{t\ge 0}$ is decomposed as $Q = \Gamma\Gamma^*$ where $\Gamma \in \gamma(H, V)$. Hence, the mappings

$$W_H : \Gamma^* v* \to \langle W(t), v^* \rangle \quad \text{for } v^* \in V^*,\ t \ge 0$$

uniquely extend to a H-cylindrical Wiener process $W_H(t)$. Conversely, when V is a separable Hilbert space, $W(t)$ is a V^*-cylindrical Wiener process by defining

$$W(t)v^* := W_H(\Gamma^* v^*) \quad \text{for } v^* \in V^*,\ t \ge 0.$$

We have the following theorems that concern the representations of V-cylindrical Wiener process and V-valued Wiener process.

Theorem 2.18. *[38, Theorem 7.4] For a V-cylindrical Wiener process $(W_V(t))_{t\ge 0}$, there exists a separable Hilbert space H with an orthonormal basis $(h_n)_{n\in\mathbb{N}}$, a bounded linear operator $\Gamma \in L(H, V^*)$ and a sequence of independent standard normal random variables $(\xi_n)_{n\in\mathbb{N}}$ such that*

$$W_V(t)v = \sum_{k=1}^{\infty} \langle \Gamma h_k, v \rangle \xi_k \quad \text{in } L^2(\Omega) \text{ for all } v \in V.$$

By setting $V = H$ and $\Gamma = \mathrm{Id}$ in Theorem 2.18, a H-cylindrical Wiener process $(W_H(t))_{t\ge 0}$ is given by

$$W_H(t)h = \sum_{k=1}^{\infty} (h_k, h)_H \xi_k(t), \quad \text{for all } h \in H. \tag{2.4}$$

Hence the covariance operator Q of W_H is a mapping: $H \to H$. This is how a cylindrical Wiener process and its covariance operator is defined on Hilbert space in most reference.

The following theorem gives the representation of a V-valued Wiener process with respect to a γ-radonifying operator.

Theorem 2.19. *[38, Theorem 7.4] For an adapted V-valued Wiener process $(W(t))_{t\ge 0}$*

the following are equivalent:

1. $(W(t))_{t\geq 0}$ *is an* V*-valued Wiener process;*
2. *there exists a separable Hilbert space* H *with an orthonormal basis* $(h_n)_{n\in\mathbb{N}}$*, a* γ*-radonifying operator* $\Gamma \in \gamma(H, V)$ *and a sequence of independent real-valued Wiener processes* $(\xi_n(t))_{n\in\mathbb{N}}$ *such that*

$$W(t) = \sum_{k=1}^{\infty} \Gamma h_k \xi_k(t) \quad \text{in } L^2(\Omega; V) \tag{2.5}$$

By the series representation in (2.4) and (2.5), a V-valued Wiener process $w(t)$ can be written in the form $\Gamma W_H(t)$ with

$$w(t) = \sum_{k=1}^{\infty} \Gamma h_k W_H(t) h_k \tag{2.6}$$

for an arbitrary orthonormal basis $(h_n)_{n\in\mathbb{N}}$ of H.

2.6 Stochastic integration

With H-cylindrical Wiener process $(W_H(t))_{t\geq 0}$ defined as above, we follow [48] and [47] closely in defining the stochastic integral with respect to W_H. Let $\Phi : (0, T) \to L(H, V)$ be a $L(H, V)$-valued step function with the form $\Phi(t) = \mathbf{1}_{(a,b)} \otimes (h \otimes v)$ with $0 \leq a < b \leq T$, $h \in H$ and $v \in V$, where $h \otimes v$ denotes the operator in $L(H, V)$ given by

$$(h \otimes v)h' = (h, h')_H v, \quad \text{for } h' \in H. \tag{2.7}$$

The random variable $\int_0^T \Phi(t) dW_H(t)$ is defined as following:

$$\int_0^T \Phi(t) dW_H(t) := (W_H(b)h - W_H(a)h) \otimes v. \tag{2.8}$$

We recall that the *finite rank operators* are the ones in $L(H, V)$ as the linear combination of operators in (2.7). Hence the random variable in (2.8) can be extended to all the step functions taking value in the finite rank operators in $L(H, V)$ by linearity. We call such functions *finite rank step functions*.

Let $L^2(0,T;H)$ denote the space of the functions $f:(0,T)\to H$ with the norm $\|f\|_{L^2(0,T;H)}=\left(\int_0^T\|f\|_H^2\right)^{\frac{1}{2}}$. It is obvious that any step function Φ uniquely defines a bounded operator $\Gamma_\Phi\in L(L^2(0,T;H),V)$ by

$$\Gamma_\Phi f:=\int_0^T\Phi(t)f(t)dt,\quad \text{for } f\in L^2(0,T;H).$$

We have the next theorem regarding the Itô isometry.

Theorem 2.20 (Itô isometry)**.** *[47, Theorem 6.14] For all finite rank step functions $\Phi:(0,T)\to L(H,V)$, we have $\Gamma_\Phi\in\gamma(L^2(0,T;H),V)$, the stochastic integral $\int_0^T\Phi dW_H$ is a Gaussian random variable, and*

$$\mathbf{E}\left\|\int_0^T\Phi(t)dW_H(t)\right\|^2=\mathbf{E}\|\Gamma_\Phi\|^2_{\gamma(L^2(0,T;H),V)}.$$

We are able to define the *stochastically integrable* functions and consequently *stochastic integral* in the following statement.

Definition 2.21. A function $\Phi:(0,T)\to L(H,V)$ is said to be *stochastically integrable* with respect to H-cylindrical Wiener process W_H if there exits a sequence of finite rank step functions $\Phi_n:(0,T)\to L(H,V)$ such that:

1. for all $h\in H$ we have $\lim_{n\to\infty}\Phi_n h=\Phi h$ in measure;
2. there exists a V-valued random variable U such that

$$\lim_{n\to\infty}\int_0^T\Phi_n(t)dW_H(t)=U$$

   in probability.

The *stochastic integral* of a stochastically integrable function $\Phi:(0,T)\to L(H,V)$ is then defined as the limit in probability

$$\int_0^T\Phi(t)dW_H(t):=\lim_{n\to\infty}\int_0^T\Phi_n(t)dW_H(t).$$

With Theorem 2.19 and Definition 2.23, we have the extension of Itô isometry to the general H-strongly measurable functions Φ.

Theorem 2.22. *[47, Theorem 6.17] Let W_H be a H-cylindrical Wiener process. For an H-strongly measurable function $\Phi : (0,T) \to L(H,V)$, the followings are equivalent:*

1. *Φ is stochatically integrable with respect to W_H;*

2. *Let Φ^* denote the adjoint operator of Φ such that $\Phi^* v^* : (0,T) \to H$ for $v^* \in V^*$. We have $\Phi^* v^* \in L^2(0,T;H)$ for all $v* \in V^*$, and there exits a V-valued random variable U such that for all $v^* \in V^*$, we have*

$$\langle U, v* \rangle = \int_0^T \Phi^* v^*(t) dW_H(t), \ a.e;$$

3. *$\Phi^* v^* \in L^2(0,T;H)$ for all $v* \in V^*$, and there exits an operator $\Gamma \in \gamma(L^2(0,T;H),V)$ such that for all $f \in L^2(0,T;H)$ and $v^* \in V^*$ we have*

$$\langle \Gamma f, v^* \rangle = \int_0^T \langle \Phi(t) f(t), v^* \rangle dt.$$

If these conditions are satisfied, the random variable U and the operator Γ is uniquely determined, we have $U = \int_0^T \Phi(t) dW_H(t)$ a.e., and

$$\mathbf{E} \left\| \int_0^T \Phi(t) dW_H(t) \right\|^2 = \mathbf{E} \|\Gamma\|^2_{\gamma(L^2(0,T;H),V)}.$$

We finish the section with Burkholder-Davis-Gundy inequality, which is essential for the error estimates in the next chapter.

Theorem 2.23. *[48, Theorem 5.9, 5.12] Let V be a reflexive Sobolev space [1], W_H be a H-cylindrical Wiener process and $\Phi : [0,T] \times \Omega \to \gamma(H,V)$ be H-strongly measurable and adapted. If $\Phi \in L^2(0,T;\gamma(H,V))$ a.s., then Φ is stochastic integrable*

[1] The Sobolev spaces $W^{k,p}$ and Sobolev-Slobodeckij spaces $W^{s,p}$ with $p \in (1,\infty)$ are reflexive.

with respect to W_H *and*

$$\mathbf{E} \sup_{t\in[0,T]} \left\| \int_0^t \Phi(\tau) dW_H(\tau) \right\|^p \leq C_{p,U} \mathbf{E} \|\Phi\|^p_{L^2(0,T;\gamma(H,V))},$$

for all $p \in (1, \infty)$.

Chapter 3

SPDEs with boundary noise

In this chapter, we recall how to reformulate a boundary noise problem as an abstract stochastic evolution equation, show existence and uniqueness of solutions, and provide spatial and temporal regularity of these solutions. This chapter also sets up the assumptions required on our solutions for numerical approximations to hold.

In this thesis, we are considering the stochastic partial differential equations (SPDE) in a smooth convex domain $D \subseteq \mathbb{R}^n$ with boundary noise given by

$$\begin{aligned} \frac{\partial U}{\partial t}(t,x) + {} & [\mathcal{A}U(t,x) + f(t,x,U(t,x))] \\ & = \bar{b}(t,x,U(t,x))\frac{\partial w_1}{\partial t}(t,x) \quad t \in (0,T],\ x \in D, \\ \mathcal{B}u(t,x) &= \bar{c}(t,x,U(t,x))\frac{\partial w_2}{\partial t}(t,x) \quad t \in (0,T],\ x \in \partial D, \\ U(0,x) &= U_0(x) \quad x \in D, \end{aligned} \tag{3.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ is given by (1.1) and (1.3) respectively, $\{w_k\}_{k=1,2}$ are two independent Wiener processes with values in Sobolev-Slobodeckij spaces $W^{\alpha,p}(D)$ and $W^{\beta,q}(\partial D)$ with $\alpha, \beta \in [0,1)$ and $p, q \in (1,\infty)$.

3.1 Abstract formulation

We now proceed to formulate the boundary noise problem (3.1) as a stochastic evolution equation taking values in a Sobolev-Slobodeckij space. Let $W^{\alpha,p} := W^{\alpha,p}(D)$ and $\partial W^{\beta,q} := W^{\beta,q}(\partial D)$ be Sobolev-Slobodeckij spaces where $w_1(t,x)$ and $w_2(t,x)$ take values respectively. Let $H := L^2(D)$ be a Hilbert space and $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Additionally for the boundary term in (3.1) being normal trace in the sense of Proposition 2.8, $\bar{c}(t,x,U(t,x))\frac{\partial w_2}{\partial t}(t,x)$ takes value in $V := W^{s-1-\frac{1}{p},p}(\partial D)$ where $s \in (1, 1+\frac{1}{p}), p \in [2,\infty)$, which is defined via duality, e.g. (5.16) in [2].

Rather than treating $U, f, \bar{b}$ and $\bar{c}$ as functions of x, we consider them as mappings, e.g., $f(t,x,U(t,x)) = f(t,U(t))(x)$. Hence we have the terms of (3.1) in map as follows:

$$\begin{aligned} f(t,U(t)) &:= f(t,U(t))(x) : [0,T] \times \Omega \times H \to H; \\ \bar{b}(t,U(t)) &:= \bar{b}(t,U(t))(x) : [0,T] \times \Omega \times H \to L(W^{\alpha,p}, H); \\ \bar{c}(t,U(t)) &:= \bar{c}(t,U(t))(x) : [0,T] \times \Omega \times H \to L(\partial W^{\beta,q}, V); \\ U(t) &:= U(t)(x) : [0,T] \times \Omega \to H. \end{aligned}$$

We note that functions $f, \bar{b}$ in (3.1) are jointly measurable and adapted stochastic processes such that the mappings $f, \bar{b}$ above are strongly measurable and adapted. Moreover, the mapping $\bar{c}$ is strongly measurable and adapted.

Let $(W_{H_1}(t))_{t\geq 0}$ and $(W_{H_2}(t))_{t\geq 0}$ be H_1- and H_2-cylindrical Wiener processes, where H_1 and H_2 are two separable Hilbert spaces. We introduce the γ-radonifying operators $\Gamma_1 \in \gamma(H_1, W^{\alpha,p})$ and $\Gamma_2 \in \gamma(H_2, \partial W^{\beta,q})$ with $p,q \geq 2$ such that w_1 and w_2 are induced by W_{H_1} and W_{H_2} uniquely of the form

$$w_1(t) = \Gamma_1 W_{H_1}(t), \; w_2(t) = \Gamma_2 W_{H_2}(t).$$

We further denote

$$b(t,U(t)) := \bar{b}(t,U(t))\Gamma_1, \quad c(t,U(t)) := \bar{c}(t,U(t))\Gamma_2.$$

With Proposition 2.14, the operator $b(t,U(t))$ and $c(t,U(t))$ are γ-radonifying. We note that the functions $f, \bar{b}$ and $\bar{c}$ in (3.1) are strongly measurable and adapted stochastic processes such that the mappings f, b and c are strongly measurable and adapted.

We assume that the boundary ∂D belongs to C^2 so that the normal traces of U are well defined in the Sobolev-Slobedeckij sense. Consider a deterministic elliptic equation on the domain D given by

$$\begin{aligned} (\delta + \mathcal{A})u &= 0 \quad \text{in } D \\ \mathcal{B}u &= g \quad \text{in } \partial D. \end{aligned}$$

where $\delta > 0$ is arbitrary. For $g \in V$, the problem has a unique solution

$u = v + N_\delta g \in W^{s,p}(D)$, where $v \in W^{s,p}(D)$ is the unique solution to the homogeneous boundary problem

$$\begin{aligned}(\delta + A)v &= -(\delta + A)N_\delta g \quad \text{in } D\\ \mathcal{B}v &= 0 \quad \text{in } \partial D.\end{aligned}$$

The operator A is a mapping such that $Av = \mathcal{A}v$ in D with $\mathcal{B}u = 0$ in ∂D. The formula (9.15) in [2] implies that $N_\delta \in L(V, W^{s,p})$

Let $H^s := W^{s,2}(D)$ be a Hilbert space, we define the operator $A : \mathcal{D}(A) \subset H \to H$ by $Ax = \mathcal{A}x$ and

$$\dot{H}^2 := \mathcal{D}(A) = \{u \in H^2 : \mathcal{B}u = 0\}. \tag{3.2}$$

Since $(\delta + A)N_\delta \in L(\partial W^{\beta,q}, W^{s-2,p})$, the mapping

$$(\delta + A)N_\delta c(t, U(t)) \in \gamma(H_2, W^{s-2,p}). \tag{3.3}$$

Note that in the case $p = q = 2$, then the conditions $b \in \gamma(H_1, H)$ and $(\delta + A)N_\delta c(t, U(t)) \in \gamma(H_2, H)$ are equivalent to saying that b and $(\delta + A)N_\delta c(t, U(t))$ are Hilbert-Schmidt. We denote $\|\cdot\|_{L_2} := \|\cdot\|_{L_2(H_*,H)}$ when the Hilbert spaces H_* and H are clear within the context for the rest of the thesis.

Now (3.1) is posed as the abstract evolution equation given by

$$\begin{aligned}dU(t) + [AU(t) + f(t, U(t))]\,dt &= b(t, U(t))dW_{H_1}(t)\\ &\quad + (\delta + A)N_\delta c(t, U(t))dW_{H_2}(t),\\ U(0) &= U_0.\end{aligned} \tag{3.4}$$

The definition of operator A in (3.2) implies that (3.4) has a homogeneous Neumann boundary condition.

To summarize the above conditions with respect to the term f, b and c as well as to present the necessary conditions for the uniqueness and existence of the solution to (3.4), we have the following assumptions:

Assumption 3.1. *The linear operator $A : \mathcal{D}(A) \subset H \to H$ is densely defined, self-adjoint and positive definite with compact inverse.*

Assumption 3.2. *The mapping $f : [0, T] \times \Omega \times H \to \dot{H}^{-1}$ is strongly measurable and adapted. The function f has linear growth and is Lipschitz continuous in space* *uniformly in $[0, T] \times \Omega$; that is, there exists a constant C such that*

$$\|f(t, \omega, h_1) - f(t, \omega, h_2)\|_{-1} \leq C\|h_1 - h_2\| \tag{3.5}$$

for all $h_1, h_2 \in H$, also, there exists a constant C such that

$$\|f(t_1, \omega, h) - f(t_2, \omega, h)\|_{-1} \leq C(1 + \|h\|)(t_2 - t_1)^{\frac{r}{2}} \tag{3.6}$$

for all $h \in H$, $0 \leq t_1 \leq t_2 \leq T$.

Assumption 3.3. *The mapping $b : [0, T] \times \Omega \times H \to L_2(H_1, H)$ is strongly measurable and adapted. The function b has linear growth and is Lipschitz continuous in space uniformly in $[0, T] \times \Omega$; that is, there exists a constant C such that*

$$\|b(t, \omega, h_1) - b(t, \omega, h_2)\|_{L_2} \leq C\|h_1 - h_2\| \tag{3.7}$$

for all $h_1, h_2 \in H$, also, there exists a constant C such that

$$\|b(t_1, \omega, h) - b(t_2, \omega, h)\|_{L_2} \leq C(1 + \|h\|)(t_2 - t_1)^{\frac{r}{2}} \tag{3.8}$$

for all $h \in H$, $0 \leq t_1 \leq t_2 \leq T$.

Assumption 3.4. *The mapping $(\delta + A)N_\delta c : [0, T] \times \Omega \times H \to L_2(H_2, H)$ is strongly measurable and adapted. The function $(\delta + A)N_\delta c$ has linear growth and is Lipschitz continuous in space uniformly in $[0, T] \times \Omega$; that is, there exists a constant C such that*

$$\|(\delta + A)N_\delta \left(c(t, \omega, h_1) - c(t, \omega, h_2)\right)\|_{L_2} \leq C\|h_1 - h_2\| \tag{3.9}$$

for all $h_1, h_2 \in H$, also, there exists a constant C such that

$$\|(\delta + A)N_\delta \left(c(t_1, \omega, h) - c(t_2, \omega, h)\right)\|_{L_2} \leq C(1 + \|h\|)(t_2 - t_1)^{\frac{r}{2}} \tag{3.10}$$

for all $h \in H$, $0 \leq t_1 \leq t_2 \leq T$.

Assumption 3.5. *The initial value $U_0 : \Omega \to \dot{H}^r$ is strongly measurable and for some $p \geq 2$ it holds that*

$$\|U_0\|_{L^p(\Omega; \dot{H}^r)} < \infty.$$

Given the assumptions and conditions above, the notion of the mild solution to (3.4) is given by

Definition 3.6. Let $p \geq 2$, $(E(t))_{t\geq 0}$ be C_0-semigroup on H and its infinitesimal generator be $-A$. A càdlàg and adapted stochastic process $U : [0, T] \times \Omega \to H$ is called a p-fold integrable *mild solution* to (3.4) if

$$\sup_{t\in[0,T]} \|U(t)\|_{L^p(\Omega;H)} < \infty,$$

and for all $t \in [0, T]$, it holds that

$$\begin{aligned} U(t) = E(t)U_0 - \int_0^t E(t-s)f(s, U(s))ds + \int_0^t E(t-s)b(s, U(s))dW_{H_1}(s) \\ + \int_0^t E(t-s)(\delta + A)N_\delta c(s, U(s))dW_{H_2}(s). \end{aligned} \tag{3.11}$$

Lemma 3.7. *[30, Lemma 2.26] Under Assumption 3.2, 3.3 and 3.4, there exists a constant C such that*

$$\begin{aligned} &\|f(t_1, Y(t_1)) - f(t_2, Z(t_2))\|_{L^p(\Omega;\dot{H}^{-1})} \\ &\qquad \leq C\left(1 + \|Y(t_1)\|_{L^p(\Omega;H)}\right)|t_1 - t_2|^{\frac{r}{2}} + C\|Y(t_1) - Z(t_2)\|_{L^p(\Omega;H)}, \end{aligned}$$

$$\begin{aligned} &\|b(t_1, Y(t_1)) - b(t_2, Z(t_2))\|_{L^p(\Omega;L_2)} \\ &\qquad \leq C\left(1 + \|Y(t_1)\|_{L^p(\Omega;H)}\right)|t_1 - t_2|^{\frac{r}{2}} + C\|Y(t_1) - Z(t_2)\|_{L^p(\Omega;H)}, \end{aligned}$$

and

$$\begin{aligned} &\|(\delta + A)N_\delta\left(c(t_1, Y(t_1)) - c(t_2, Z(t_2))\right)\|_{L^p(\Omega;L_2)} \\ &\qquad \leq C\left(1 + \|Y(t_1)\|_{L^p(\Omega;H)}\right)|t_1 - t_2|^{\frac{r}{2}} + C\|Y(t_1) - Z(t_2)\|_{L^p(\Omega;H)} \end{aligned}$$

for all $t_1, t_2 \in [0, T]$.

3.2 Existence and uniqueness of a mild solution

We have the following result with regard to the existence and uniqueness of the solution to (3.4).

Theorem 3.8. *[30, Theorem 2.25][9, Lemma 3.1] Under Assumption 3.1 - 3.5, for* $p \in [2, \infty)$*, there exists a unique p-fold integrable mild solution* $U : [0, T] \times \Omega \to H$ *to* (3.4) *such that for every* $t \in [0, T]$ *and every* $r \in [0, 1)$*, it holds that* $\mathbf{P}(U(t) \in \dot{H}^r) = 1$ *with*

$$\sup_{t \in [0,T]} \|U(t)\|_{L^p(\Omega; \dot{H}^r)} < \infty.$$

Note that U is the unique solution to (3.1) since the trace of U is well defined under Assumption 3.4.

3.3 Regularity of solutions

The spatial regularity of the solution is implicit in Theorem 3.8. In addition, we have the temporal regularity given by the following Theorem.

Theorem 3.9. *[39, Theorem 4.14] Let* $U(t) \in L^p(\Omega; \dot{H}^r)$ *with* $r \in [0, 1), t_1, t_2 \in [0, T]$ *be the unique mild solution to* (3.4)*, under Assumption 3.2 - 3.4, it holds that*

$$\left(\mathbf{E}\|U(t_1) - U(t_2)\|_s^p\right)^{\frac{1}{p}} \leq C|t_1 - t_2|^{\beta}$$

where $0 \leq s < \frac{r}{2}$ *and* $0 < \beta \leq \frac{r-s}{2}$.

Chapter 4

Error estimates for the Galerkin approximation

In this chapter, we prove our main results regarding approximation of the Neumann boundary noise problem (3.1) using a Galerkin approximation approach.

4.1 Spatial semi-discretization

We start with a review of the Galerkin finite element method for the spatial semi-discretization of the Hilbert space $H := L^2(D)$. Let $\dot{H}^s = \mathcal{D}(A^{\frac{s}{2}})$ and the operator A satisfy Assumption 3.1. First we introduce the finite element subspaces $(S_h)_{h\in(0,1]} \subset \dot{H}^1$ as in [10]. We assume S_h have the following properties:

Assumption 4.1. *Let $(S_h)_{h\in(0,1]}$ be a family of finite dimensional subspaces of $\dot{H}^1$ such that*

$$\inf_{x_h\in S_h} \|x_h - x\|_s \to 0 \quad \text{as } h \to 0 \ \forall x \in \dot{H}^s(D), s = \{-1, 0, 1\},$$

and additionally there exists a constant C independent of h such that

$$\inf_{x_h\in S_h} \|x_h - x\| \leq Ch^s\|x\|_s, \quad \forall x \in \dot{H}^s(D), s \in [0, 2].$$

Let $R_h : \dot{H}^1 \to S_h$ denote the orthogonal projector with respect to the inner product $(\cdot,\cdot)_1 := (A^{\frac{1}{2}}\cdot, A^{\frac{1}{2}}\cdot)$ in $\dot{H}^1$, and $P_h : \dot{H}^{-1} \to S_h$ denote the generalized orthogonal projector. Hence for each $x \in \dot{H}^1$,

$$(R_h x, y_h)_1 = (x, y_h)_1, \quad \forall y_h \in S_h. \tag{4.1}$$

and for each $x \in \dot{H}^{-1}$,

$$(P_h x, y_h) = \langle x, y_h \rangle, \quad \forall y_h \in S_h, \tag{4.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pair between $\dot{H}^1$ and $\dot{H}^{-1}$ [1].

The following proposition states that the projector $P_h x$ is the best approximation in L^2-norm to $x \in L^2(D)$ and the *quasi-best* approximation in $\dot{H}^1$-norm to $x \in \dot{H}^1(D)$, i.e.,

$$\|x - P_h x\|_1 \leq C \inf_{y_h \in S_h} \|x - y_h\|_1.$$

Proposition 4.2. *[10, Proposition 2.1] Let $R_h : \dot{H}^1 \to S_h$ be defined by* (4.1) *and $P_h : \dot{H}^{-1} \to S_h$ be defined by* (4.2). *There exists a constant C independent of h such that*

$$\|x - P_h x\| \leq \|x - R_h x\|, \quad \forall x \in L^2(D),$$

and

$$\|x - P_h x\|_1 \leq C \|x - R_h x\|_1, \quad \forall x \in \dot{H}^1(D).$$

Next we introduce the discrete version of A denoted by $A_h : S_h \to S_h$. For $x_h \in S_h$, the operator A_h is defined as

$$(x_h, y_h)_1 = (A_h x_h, y_h), \quad \forall y_h \in S_h. \tag{4.3}$$

It is observed that

$$(A_h x_h, y_h) = (x_h, y_h)_1 = (x_h, A_h y_h), \quad \forall x_h, y_h \in S_h$$

and

$$(A_h x_h, x_h) = (x_h, x_h)_1 = \|x_h\|_1^2 \geq 0, \quad \forall x_h \in S_h.$$

Hence the operator A_h is self-adjoint and positive-definite on S_h. Consequently, $-A_h$ is the generator of an analytic semigroup on S_h, which is denoted by $E_h(t) := e^{-A_h t}$. The semigroup $(E_h(t))_{t \geq 0}$ has the similar smoothing

[1] See Theorem 2.4

property to Lemma 2.5 (i), for $\rho \geq 0$ and $y_h \in S_h$,

$$\|A_h^{\rho} E_h(t) y_h\| \leq C t^{-\rho} \|y_h\|, \quad \forall t > 0. \tag{4.4}$$

As a result of (4.4), we have the estimate

$$\|E_h(t)P_h x\| = \|A_h^{\frac{1}{2}} E_h(t) A_h^{-\frac{1}{2}} P_h x\| \leq C t^{-\frac{1}{2}} \|A_h^{-\frac{1}{2}} P_h x\| \leq C t^{-\frac{1}{2}} \|x\|_{-1} \tag{4.5}$$

for all $x \in \dot{H}^{-1}$, $t > 0$ and $h \in (0, 1]$.

4.2 Error estimates for the spatially semi-discrete approximation

We recall that for a deterministic linear initial value problem

$$X'(t) + AX(t) = 0, \quad t > 0, \quad \text{with } X(0) = x, \tag{4.6}$$

the unique solution is given by

$$X(t) = E(t)x,$$

(See [51]).

Applying the Galerkin FEM to (4.6), we have the semi-discrete equation

$$X_h'(t) + A_h X_h(t) = 0, \quad t > 0, \quad \text{with } X_h(0) = P_h x, \tag{4.7}$$

with the solution given by

$$X_h(t) = E_h(t) P_h x.$$

Hence, the error between the discrete solution $X_h(t)$ and the continuous solution $X(t)$ is

$$\|X_h(t) - X(t)\| = \|\left(E_h(t)P_h - E(t)\right)x\|.$$

The next two lemmas are concerned with the error operator $F_h := E_h(t)P_h - E(t)$ given non-smooth initial values $x \in \dot{H}^s$ with $s \in [-1, 1)$.

Lemma 4.3. *Under Assumption 4.1, the operator F_h has the following estimates:*

i) *[45, Theorem 3.5] Let $0 \leq \nu \leq \mu \leq 1$, there exists a constant C such that*

$$\|F_h(t)x\| \leq Ch^{\mu} t^{-\frac{\mu-\nu}{2}} \|x\|_{\nu} \ \ \forall x \in \dot{H}^{\nu}, \ t > 0, \ h \in (0,1].$$

ii) *Let $0 \leq \mu \leq 1$ and $0 \leq \nu \leq 1$, there exists a constant C such that*

$$\|F_h(t)x\| \leq Ch^{\mu} t^{-\frac{\mu+\nu}{2}} \|x\|_{-\nu} \ \ \forall x \in \dot{H}^{-\nu}, \ t > 0, \ h \in (0,1].$$

Proof. For (i), we refer to the proof of [45, Theorem 3.5].

For (ii), We have

$$\begin{aligned} \|F_h(t)x\| &\leq \|E_h(t)P_h x - P_h E(t)x\| + \|(P_h - \mathrm{Id})E(t)x\| \\ &=: \Theta(t) + \Pi(t) \end{aligned}$$

We estimate Θ by constructing a new initial value problem with subtracting (4.7) from (4.6):

$$\begin{aligned} &X'(t) - P_h X'(t) + P_h X'(t) - X_h'(t) \\ &\qquad + AX(t) - AP_h X(t) + AP_h X(t) - A_h X_h(t) = 0 \\ &X(0) - P_h X(0) = x - P_h x. \end{aligned} \tag{4.8}$$

With the definition of A_h in (4.3), the problem (4.8) is written as,

$$\begin{aligned} \Theta' &= A\Theta(t) - \Pi'(t), \\ \Theta(0) &= (P_h - \mathrm{Id})x, \end{aligned}$$

with the solution

$$\begin{aligned} \Theta(t) &= E(t)(P_h - \mathrm{Id})x - \int_0^t E(t-s)A\Pi(s)ds \\ &= \Pi(t) - \int_0^t E(t-s)E(s)(P_h - \mathrm{Id})Axds \\ &= \Pi(t) - (P_h - \mathrm{Id})tAE(t)x \end{aligned}$$

Hence,

$$\|F_h(t)x\| \leq 2\|\Pi(t)\| + C\|(P_h - \mathrm{Id})tE(t)Ax\|.$$

Given P_h is the best approximation in the L^2-norm and the quasi-best approximation in the $\dot{H}^1$-norm (Proposition 4.2), we have

$$\|F_h(t)x\| \leq Ch^\mu \|E(t)x\|_\mu + Ch^\mu t \|E(t)Ax\|_\mu, \quad \mu \in \{0,1\}.$$

The intermediate cases are obtained by the interpolation technique demonstrated in [45, Theorem 3.5] [2]. Then we get

$$\|E(t)x\|_\mu = \|A^{\frac{\mu+\nu}{2}} E(t) A^{-\frac{\nu}{2}} x\|. \tag{4.9}$$

With the smoothing property of $E(t)$ shown in Lemma 2.5 (i), the proof is completed. □

Lemma 4.4. *Under Assumption 4.1 the operator F_h has the following estimates:*

i) Let $0 \leq \mu \leq 1$, $0 \leq \nu \leq 1$ and $\mu + \nu < 1$, there exists a constant C such that

$$\left(\int_0^t \|F_h(s)x\|^2 ds \right)^{\frac{1}{2}} \leq Ch^\mu \|x\|_{-\nu} \ \ \forall x \in \dot{H}^{-\nu}, \ t > 0, \ h \in (0,1].$$

ii) [30, Lemma 3.9 (i)] Let $0 \leq \nu \leq 1$, there exists a constant C such that

$$\left\| \int_0^t F_h(s)x ds \right\| \leq Ch^{2-\nu} \|x\|_{-\nu} \ \ \forall x \in \dot{H}^{-\nu}, \ t > 0, \ h \in (0,1].$$

Proof. For (i), it is enough to show the proof for $\mu = 0$ and $\mu = 1$. The intermediate cases are acquired by the same interpolation technique demonstrated in [45, Theorem 3.5]. From the proof of Lemma 4.3 (ii) we have

$$\left(\int_0^t \|F_h(s)x\|^2 ds \right)^{\frac{1}{2}} \leq Ch^\mu \left(\int_0^t \|E(s)x\|_\mu^2 ds \right)^{\frac{1}{2}}.$$

Thus, by Lemma 2.5 (iii) and (4.9) we complete the proof.

For (ii), we refer to the proof of [30, Lemma 3.9 (i)]. □

Besides the assumptions in Chapter 3, another assumption with regards to the γ-radonifying $(\delta + A)N_\delta c$ is necessary for the optimal convergence result.

[2] The details are shown in Appendix A.3

Assumption 4.5. *Let $\theta \in [0,1)$ and $\theta + r < 1$. For all $h \in \dot{H}^r$, the mapping $A^{-\frac{\theta}{2}}(\delta + A)N_\delta c : [0,T] \times \Omega \times H \to L_2$ is strongly measurable and adapted. The function $A^{-\frac{\theta}{2}}(\delta + A)N_\delta c$ has linear growth and is Lipschitz continuous in space uniformly in $[0,T] \times \Omega$; that is, there exists a constant C such that*

$$\|A^{-\frac{\theta}{2}}(\delta + A)N_\delta \left(c(t,\omega,h_1) - c(t,\omega,h_2)\right)\|_{L_2} \le C\|h_1 - h_2\|_r$$

for all $h_1, h_2 \in \dot{H}^r$, also, there exists a constant C such that

$$\|A^{-\frac{\theta}{2}}(\delta + A)N_\delta c(t,\omega,h)\|_{L_2} \le C(1 + \|h\|_r)$$

for all $h \in \dot{H}^r$.

With the operator A_h and the projector P_h defined above, the semi-discrete equation of the continuous problem (3.4) is given by

$$\begin{aligned} dU_h(t) + [A_h U_h(t) + P_h f(t, U_h(t))]\, dt &= P_h b(t, U_h(t)) dW_{H_1}(t) \\ &\quad + P_h(\delta + A)N_\delta c(t, U_h(t)) dW_{H_2}(t), \\ U_h(0) &= P_h U_0. \end{aligned} \tag{4.10}$$

As the same for (3.4), there exists a stochastic process $U_h(t) : [0,T] \times \Omega \to S_h$ as the unique solution to (4.10) given by

$$\begin{aligned} U_h(t) = E_h(t)P_h U_0 &- \int_0^t E_h(t-s)P_h f(s, U_h(s))ds \\ &+ \int_0^t E_h(t-s)P_h b(s, U_h(s))dW_{H_1}(s) \\ &+ \int_0^t E_h(t-s)P_h(\delta + A)N_\delta c(s, U_h(s))dW_{H_2}(s). \end{aligned} \tag{4.11}$$

Then we have the result with respect to the estimate of $\|U_h(t) - U(t)\|_{L^p(\Omega;H)}$ as follows:

Theorem 4.6. *Under the assumptions in Chapter 3, Assumption 4.1 and Assumption 4.5, for $r \in [0,1)$ and $p \in [2,\infty)$, there exists a constant C, independent of*

$h \in (0,1]$, *such that*

$$\|U_h(t) - U(t)\|_{L^p(\Omega;H)} \leq Ch^r, \quad \forall t \in (0,T],$$

where $U(t)$ *and* $U_h(t)$ *are the mild solutions to (3.4) and (4.10) respectively.*

Proof. With (3.11) and (4.11), we have

$$\begin{aligned}
\|U_h(t) - U(t)\|_{L^p(\Omega;H)} \leq & \|F_h(t)U_0\|_{L^p(\Omega;H)} \\
& + \left\| \int_0^t E_h(t-s)P_h f(s, U_h(s))ds \right. \\
& \left. - \int_0^t E(t-s)f(s,U(s))ds \right\|_{L^p(\Omega;H)} \\
& + \left\| \int_0^t E_h(t-s)P_h b(s, U_h(s))dW_{H_1}(s) \right. \\
& \left. - \int_0^t E(t-s)b(s,U(s))dW_{H_1}(s) \right\|_{L^p(\Omega;H)} \\
& + \left\| \int_0^t E_h(t-s)P_h(\delta + A)N_\delta c(s, U_h(s))dW_{H_2}(s) \right. \\
& \left. - \int_0^t E(t-s)(\delta + A)N_\delta c(s,U(s))dW_{H_2}(s) \right\|_{L^p(\Omega;H)}
\end{aligned} \tag{4.12}$$

The first term is the direct result of Lemma 4.3 (i) as

$$\|F_h(t)U_0\|_{L^p(\Omega;H)} \leq Ch^r \|A^{\frac{r}{2}} U_0\|_{L^p(\Omega;H)} \tag{4.13}$$

when $\mu = \nu = r$.

The second term is evaluated by two additional terms

$$\begin{aligned}
&\left\| \int_0^t E_h(t-s)P_h f(s, U_h(s))ds - \int_0^t E(t-s)f(s, U(s))ds \right\|_{L^p(\Omega;H)} \\
&\leq \left\| \int_0^t E_h(t-s)P_h \left(f(s, U_h(s)) - f(s, U(s)) \right) ds \right\|_{L^p(\Omega;H)} \\
&\quad + \left\| \int_0^t F_h(t-s)f(s, U(s))ds \right\|_{L^p(\Omega;H)} \\
&=: I_1 + I_2.
\end{aligned}$$

First by (4.5) and Lemma 3.7 we have

$$\begin{aligned}
I_1 &\leq \int_0^t \|E_h(t-s)P_h \left(f(s, U_h(s)) - f(s, U(s)) \right)\|_{L^p(\Omega;H)} \, ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|f(s, U_h(s)) - f(s, U(s))\|_{L^p(\Omega;\dot{H}^{-1})} \, ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|U_h(s) - U(s)\|_{L^p(\Omega;H)} \, ds.
\end{aligned} \tag{4.14}$$

Then the term I_2 is estimated by Lemma 4.3 (ii) with $\mu = r$ and $\nu = 1$, and

Lemma 3.7, which gives

$$
\begin{aligned}
I_2 &\le \int_0^t \|F_h(t-s)f(s,U(s))\|_{L^p(\Omega;H)}\,ds \\
&\le Ch^r \int_0^t (t-s)^{-\frac{r+1}{2}} \|f(s,U(s))\|_{L^p(\Omega;\dot{H}^{-1})}\,ds \\
&\le Ch^r \int_0^t (t-s)^{-\frac{r+1}{2}} \|f(s,U(s)) - f(0,0)\|_{L^p(\Omega;\dot{H}^{-1})}\,ds \\
&\quad + Ch^r \int_0^t (t-s)^{-\frac{r+1}{2}} \|f(0,0)\|_{L^p(\Omega;\dot{H}^{-1})}\,ds \\
&\le Ch^r \int_0^t (t-s)^{-\frac{r+1}{2}} ds \left(1 + \sup_{s\in[0,T]} \|U(s)\|_{L^p(\Omega;H)}\right) + CT^{\frac{1-r}{2}} h^r,
\end{aligned}
$$

since $\|f(0,0)\|_{L^p(\Omega;\dot{H}^{-1})} < \infty$ by Assumption 3.2. Theorem 3.8 indicates that the integral in the RHS of the estimate is finite. Hence we have

$$
I_2 \le CT^{\frac{1-r}{2}} h^r \le Ch^r. \tag{4.15}
$$

Together (4.14) and (4.15) yields

$$
\begin{aligned}
&\left\| \int_0^t E_h(t-s)P_h f(s,U_h(s))ds - \int_0^t E(t-s)f(s,U(s))ds \right\|_{L^p(\Omega;H)} \\
&\quad \le Ch^r + C\int_0^t (t-s)^{-\frac{1}{2}} \|U_h(s) - U(s)\|_{L^p(\Omega;H)}\,ds.
\end{aligned} \tag{4.16}
$$

By Theorem 2.23, the third term of the stochastic integral has the inequality

$$\left\| \int_0^t E_h(t-s)P_h b(s,U_h(s))dW_{H_1}(s) - \int_0^t E(t-s)b(s,U(s))dW_{H_1}(s) \right\|_{L^p(\Omega;H)}$$
$$\leq C \left(\mathbf{E}\left[\left(\int_0^t \|E_h(t-s)P_h b(s,U_h(s)) - E(t-s)b(s,U(s))\|_{L_2}^2 \, ds \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}.$$

The RHS is estimated by three additional terms

$$\left(\mathbf{E}\left[\left(\int_0^t \|E_h(t-s)P_h b(s,U_h(s)) - E(t-s)b(s,U(s))\|_{L_2}^2 \, ds \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}$$
$$\leq \left\| \left(\int_0^t \|E_h(t-s)P_h \left(b(s,U_h(s)) - b(s,U(s))\right)\|_{L_2}^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;\mathbb{R})}$$
$$+ \left\| \left(\int_0^t \|F_h(t-s)\left((b(s,U(s)) - b(t,U(t))\right)\|_{L_2}^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;\mathbb{R})}$$
$$+ \left\| \left(\int_0^t \|F_h(t-s)b(t,U(t))\|_{L_2}^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;\mathbb{R})}$$
$$=: I_3 + I_4 + I_5.$$

First we estimate the term I_3 by (4.4) with $\rho = 0$ and Lemma 3.7, which indicates

$$
\begin{aligned}
I_3 &\le C \left\| \left(\int_0^t \|b(s, U_h(s)) - b(s, U(s))\|_{L_2}^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;\mathbb{R})} \\
&\le C \left(\int_0^t \|b(s, U_h(s)) - b(s, U(s))\|_{L^p(\Omega;L_2)}^2 \, ds \right)^{\frac{1}{2}} \\
&\le C \left(\int_0^t \|U_h(s) - U(s)\|_{L^p(\Omega;H)}^2 \, ds \right)^{\frac{1}{2}}.
\end{aligned}
\tag{4.17}
$$

Next by applying Lemma 4.3 (i) with $\mu = r$ and $\nu = 0$, Lemma 3.7 and Theorem 3.9, the term I_4 is estimated as follows

$$
\begin{aligned}
I_4 &\le Ch^r \left\| \left(\int_0^t (t-s)^{-r} \|b(s, U(s)) - b(t, U(t))\|_{L_2}^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;\mathbb{R})} \\
&\le Ch^r \left(\int_0^t (t-s)^{-r} \|b(s, U(s)) - b(t, U(t))\|_{L^p(\Omega;L_2)}^2 \, ds \right)^{\frac{1}{2}} \\
&\le Ch^r \left(\int_0^t (t-s)^{2\beta} ds \right)^{\frac{1}{2}} \\
&\le \frac{C}{1+2\beta} T^{1+2\beta} h^r \le Ch^r, \quad \text{where } 0 < \beta \le \frac{r}{2}.
\end{aligned}
\tag{4.18}
$$

Finally the term I_5 is estimated by Lemma 4.4 with $\mu = r$ and $\nu = 0$, Lemma 3.7 and Theorem 3.8 as follows,

$$
\begin{aligned}
I_5 &\le Ch^r \left\| \|b\,(t, U(t))\|_{L_2} \right\|_{L^p(\Omega;\mathbb{R})} \\
&\le CT^{\frac{r}{2}} h^r \left(1 + \sup_{t\in[0,T]} \|U(t)\|_{L^p(\Omega;H)} \right) \le Ch^r.
\end{aligned}
\tag{4.19}
$$

Together by (4.17)-(4.19), we have

$$\begin{aligned}
&\left\| \int_0^t E_h(t-s)P_h b(s,U_h(s))dW_{H_1}(s) \right.\\
&\quad \left. - \int_0^t E(t-s)b(s,U(s))dW_{H_1}(s) \right\|_{L^p(\Omega;H)} \\
&\quad \le Ch^r + C\left(\int_0^t \|U_h(s)-U(s)\|^2_{L^p(\Omega;H)}\, ds \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.20}$$

Similar to the estimate of the third term, the last term in (4.12) satisfies the inequality

$$\begin{aligned}
&\left\| \int_0^t E_h(t-s)P_h(\delta+A)N_\delta c(s,U_h(s))dW_{H_2}(s) \right.\\
&\quad \left. - \int_0^t E(t-s)(\delta+A)N_\delta c(s,U(s))dW_{H_2}(s) \right\|_{L^p(\Omega;H)} \\
&\quad \le C\left(\mathbf{E}\left[\left(\int_0^t \|E_h(t-s)P_h(\delta+A)N_\delta c(s,U_h(s)) \right.\right.\right.\\
&\qquad \left.\left.\left. -E(t-s)(\delta+A)N_\delta c(s,U(s))\|^2_{L_2}\, ds \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}.
\end{aligned}$$

Hence we have the RHS dominated by three terms as follows

$$
\begin{aligned}
&\left(\mathbf{E}\left[\left(\int_0^t \|E_h(t-s)P_h(\delta+A)N_\delta c(s,U_h(s))\right.\right.\right.\\
&\quad\left.\left.\left. -E(t-s)(\delta+A)N_\delta c(s,U(s))\|_{L_2}^2\,ds\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}\\
&\quad\le \left\|\left(\int_0^t \|E_h(t-s)P_h(\delta+A)N_\delta\left[c(s,U_h(s))-c(s,U(s))\right]\|_{L_2}^2\,ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}\\
&\quad\quad+\left\|\left(\int_0^t \|F_h(t-s)(\delta+A)N_\delta\left[(c(s,U(s))-c(t,U(t))\right]\|_{L_2}^2\,ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}\\
&\quad\quad+\left\|\left(\int_0^t \|F_h(t-s)(\delta+A)N_\delta c(t,U(t))\|_{L_2}^2\,ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}\\
&=: I_6+I_7+I_8.
\end{aligned}
$$

The term I_6 and I_7 has the exactly same estimate as the term I_3 and I_4 respectively such that

$$
I_6 \le C\left(\int_0^t \|U_h(s)-U(s)\|_{L^p(\Omega;H)}^2\,ds\right)^{\frac{1}{2}}, \tag{4.21}
$$

and

$$
I_7 \le Ch^r. \tag{4.22}
$$

To estimate the term I_8, we recall the mapping property of $(\delta+A)N_\delta c : H_2 \to H^{\alpha-2}(D)$, where $\alpha > 1+r$. Let $\{\phi_m\}_{m\ge1}$ be an arbitrary orthonormal basis of the Hilbert space H_2. Then, by applying Lemma 4.4 with $\mu = r$ and $\nu = 2-\alpha$

and Theorem 3.8, under Assumption 4.5, we get

$$
\begin{aligned}
I_8 &= \left\| \left(\sum_{m=1}^{\infty} \int_0^t \|F_h(t-s)(\delta + A)N_\delta c(t, U(t))\phi_m\|^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq Ch^r \left\| \left(\sum_{m=1}^{\infty} \left\| A^{-\frac{2-\alpha}{2}} (\delta + A)N_\delta c(t, U(t))\phi_m \right\|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;\mathbb{R})} \\
&= Ch^r \left\| \left\| A^{-\frac{2-\alpha}{2}} (\delta + A)N_\delta c(t, U(t)) \right\|_{L_2} \right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq Ch^r \left(1 + \sup_{t\in[0,T]} \|U(t)\|_{L^p(\Omega;\dot{H}^r)} \right) \leq Ch^r.
\end{aligned}
\tag{4.23}
$$

The estimate of the boundary noise term results from (4.21)-(4.23) as

$$
\begin{aligned}
&\left\| \int_0^t E_h(t-s)P_h(\delta + A)N_\delta c(s, U_h(s))dW_{H_2}(s) \right. \\
&\qquad \left. - \int_0^t E(t-s)(\delta + A)N_\delta c(s, U(s))dW_{H_2}(s) \right\|_{L^p(\Omega;H)} \\
&\qquad \leq Ch^r + C \left(\int_0^t \|U_h(s) - U(s)\|^2_{L^p(\Omega;H)} \, ds \right)^{\frac{1}{2}}.
\end{aligned}
\tag{4.24}
$$

Thus by joining (4.12), (4.13), (4.16), (4.20) and (4.24), we have

$$
\begin{aligned}
\|U_h(t) - U(t)\|^2_{L^p(\Omega;H)} \leq Ch^{2r} &+ C \left(\int_0^t (t-s)^{-\frac{1}{2}} \|U_h(s) - U(s)\|_{L^p(\Omega;H)} \, ds \right)^2 \\
&+ C \int_0^t \|U_h(s) - U(s)\|^2_{L^p(\Omega;H)} \, ds.
\end{aligned}
$$

We note that by Hölder's inequality

$$\begin{aligned}
&\int_0^t (t-s)^{-\frac{1}{2}} \|U_h(s) - U(s)\|_{L^p(\Omega;H)}\, ds \\
&\quad = \int_0^t (t-s)^{-\frac{1}{4}} (t-s)^{-\frac{1}{4}} \|U_h(s) - U(s)\|_{L^p(\Omega;H)}\, ds \\
&\quad \leq \left(\int_0^t (t-s)^{-\frac{1}{2}} ds \right)^{\frac{1}{2}} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|U_h(s) - U(s)\|^2_{L^p(\Omega;H)}\, ds \right)^{\frac{1}{2}} \\
&\quad \leq \left(2T^{\frac{1}{2}} \right)^{\frac{1}{2}} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|U_h(s) - U(s)\|^2_{L^p(\Omega;H)}\, ds \right)^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \|U_h(s) - U(s)\|^2_{L^p(\Omega;H)}\, ds \\
&\quad = \int_0^t (t-s)^{\frac{1}{2}} (t-s)^{-\frac{1}{2}} \|U_h(s) - U(s)\|^2_{L^p(\Omega;H)}\, ds \\
&\quad \leq T^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|U_h(s) - U(s)\|^2_{L^p(\Omega;H)}\, ds.
\end{aligned}$$

Hence we have

$$\|U_h(t) - U(t)\|^2_{L^p(\Omega;H)} \leq Ch^{2r} + C \int_0^t (t-s)^{-\frac{1}{2}} \|U_h(s) - U(s)\|^2_{L^p(\Omega;H)}\, ds,$$

and Gronwall's Lemma (A.2) completes the proof. $\square$

4.3 Error estimates of spatio-temporally full-discrete approximation

We apply the Galerkin method and the backward Euler scheme on the deterministic homogeneous equation (4.6) to get

$$\begin{aligned} X_h^j + kA_h X_h^j &= X_h^{j-1}, \quad j = 1, 2, \dots \\ X_h^0 &= P_h x. \end{aligned}$$

It has the closed form solution

$$X_h^j = (\mathrm{Id} + kA_h)^{-j} P_h x, \quad j = 0, 1, 2, \dots. \tag{4.25}$$

We introduce the rational function

$$R(z) = \frac{1}{1+z} \quad \text{for } z \in \mathbb{R}, z \neq -1,$$

such that $R(kA_h)$ is the solution operator of (4.25) defined by

$$R(kA_h)x = \sum_{m=1}^{N_h} R(k\lambda_{h,m})(x, \phi_{h,m})\phi_{h,m}, \tag{4.26}$$

where $(\lambda_{h,m})_{m=1}^{N_h}$ be the positive eigenvalues of A_h with the corresponding orthonormal eigenvectors $(\phi_{h,m})_{m=1}^{N_h} \subset S_h$ and $\dim(S_h) = N_h$. Thus the solution (4.25) is

$$X_h^j = R(kA_h)^j P_h x, \quad j = 0, 1, 2, \dots.$$

Let the solution operator denoted by

$$E_{kh}(t) := R(kA_h)^j, \quad \text{if } t \in [t_{j-1}, t_j) \text{ for } j = 1, 2, \dots, \tag{4.27}$$

and thus the error operator denoted by

$$F_{kh}(t) := E_{kh}(t)P_h - E(t) \quad \forall t \geq 0.$$

Similar to $E_h(t)$, the operator $E_{kh}(t)$ has the smoothing property

$$\|A_h^\rho E_{kh}(t) y_h\| \leq C t_j^{-\rho} \|y_h\| \qquad \forall j = 1, 2, \dots \; y_h \in S_h$$

and consequently the inequality

$$\|E_{kh}(t)P_hx\| = \left\|A_h^{\frac{1}{2}}E_{kh}(t)A_h^{-\frac{1}{2}}P_hx\right\| \leq Ct_j^{-\frac{1}{2}}\|x\|_{-1} \leq Ct^{-\frac{1}{2}}\|x\|_{-1}, \tag{4.28}$$

for all $x \in \dot{H}^{-1}$, $h, k \in (0,1]$ and $t \in [t_{j-1}, t_j)$, $j = 1, 2, \ldots$.

Moreover, two properties of the rational function $R(z)$ hold:

$$|R(z) - e^{-z}| \leq Cz^{q+1} \qquad \text{when } q = 1, \forall z \in [0,1], \tag{4.29}$$

and

$$|R(z)| \leq e^{-cz} \qquad \forall z \in [0,1],\ c \in (0,1). \tag{4.30}$$

Hence we have the next two lemmas which are concerned with the estimates of $\|F_{kh}(t)\|$ given non-smooth initial values $x \in \dot{H}^s$ with $s \in [-1,1)$.

Lemma 4.7. *Under Assumption 4.1 the operator F_{kh} has the following estimates:*

i) [30, Lemma 3.12 (i)] Let $0 \leq \nu \leq \mu \leq 2$, there exists a constant C such that

$$\|F_{kh}(t)x\| \leq C\left(h^\mu + k^{\frac{\mu}{2}}\right)t^{-\frac{\mu-\nu}{2}}\|x\|_\nu \ \ \forall x \in \dot{H}^\nu,\ t > 0, \text{ and } h, k \in (0,1].$$

ii) Let $0 \leq \mu \leq 1$ and $0 \leq \nu \leq 1$, there exists a constant C such that

$$\|F_{kh}(t)x\| \leq C\left(h^\mu + k^{\frac{\mu}{2}}\right)t^{-\frac{\mu+\nu}{2}}\|x\|_{-\nu} \ \ \forall x \in \dot{H}^{-\nu},\ t > 0, \text{ and } h, k \in (0,1].$$

Proof. For (i), we refer to the proof of [30, Lemma 3.12 (i)].

For the proof of (ii), we follow the technique shown in [30, Lemma 3.12]. With $t \in [t_{j-1}, t_j)$, we have

$$\begin{aligned}\|F_{kh}(t)x\| \leq{}& \left\|\left(R(kA_h)^j - E_h(t_j)\right)P_hx\right\| + \left\|\left(E_h(t_j)P_h - E(t_j)\right)x\right\| \\ &+ \left\|\left(E(t_j) - E(t)\right)x\right\| =: T_1 + T_2 + T_3\end{aligned}$$

Let $(\lambda_{h,m})_{m=1}^{N_h}$ be the positive eigenvalues of A_h with the corresponding orthonormal eigenvectors $(\phi_{h,m})_{m=1}^{N_h} \subset S_h$. The term T_1 is expanded as

$$T_1^2 = \left\| \sum_{m=1}^{N_h} \lambda_{h,m}^{\frac{\nu}{2}} \left(R(k\lambda_{h,m})^j - e^{-\lambda_{h,m}t_j} \right) \left(P_h x, \lambda_{h,m}^{-\frac{\nu}{2}} \phi_{h,m} \right) \phi_{h,m} \right\|^2$$
$$= \sum_{m=1}^{N_h} \lambda_{h,m}^{\nu} \left| \left(R(k\lambda_{h,m})^j - e^{-\lambda_{h,m}kj} \right) \right|^2 \left(A_h^{-\frac{\nu}{2}} P_h x, \phi_{h,m} \right)^2.$$

When $k\lambda_{h,m} \le 1$, with (4.29), (4.30) and $\mu \le q = 1$ we have

$$\left| \left(R(k\lambda_{h,m})^j - e^{-\lambda_{h,m}kj} \right) \right| = \left| \left(R(k\lambda_{h,m}) - e^{-k\lambda_{h,m}} \right) \sum_{i=0}^{j-1} R(k\lambda_{h,m})^{j-i-1} e^{-k\lambda_{h,m}i} \right|$$
$$\le Cj(k\lambda_{h,m})^{\mu+1} e^{-c(j-1)k\lambda_{h,m}}.$$

Hence, given the fact that $\sup_{z\ge 0} z^2 e^{-cz} < \infty$ we get

$$\lambda_{h,m}^{\nu} \left| \left(R(k\lambda_{h,m})^j - e^{-\lambda_{h,m}kj} \right) \right|^2 \le C(jk)^{-(\mu+\nu)} k^{2\mu} \lambda_{h,m}^{\mu} (jk\lambda_{h,m})^{\mu+\nu+2}$$
$$e^{-\frac{\mu+\nu+2}{2} cjk\lambda_{h,m}}$$
$$\le Ct_j^{-(\mu+\nu)} k^{\mu}.$$

When $k\lambda_{h,m} > 1$ and $0 \le \mu \le 1$, we have

$$\lambda_{h,m}^{\nu} \left| \left(R(k\lambda_{h,m})^j - e^{-\lambda_{h,m}kj} \right) \right|^2 < 2k^{-\nu} (k\lambda_{h,m})^{\mu+\nu} \left(\left| R(k\lambda_{h,m})^j \right|^2 + \left| e^{-\lambda_{h,m}kj} \right|^2 \right).$$

As shown in the proof of [45, Lemma 7.3], when $k\lambda_{h,m} > 1$ and $1 \le \frac{\mu+\nu}{2} \le j$, the RHS of the inequality gives

$$(k\lambda_{h,m})^{\mu+\nu} \left| R(k\lambda_{h,m})^j \right|^2 \le \frac{(k\lambda_{h,m})^{\mu+\nu}}{(1 + ck\lambda_{h,m})^{2j}} \le Cj^{-(\mu+\nu)},$$

and

$$(k\lambda_{h,m})^{\mu+\nu} \left| e^{-\lambda_{h,m}kj} \right|^2 \le Cj^{-(\mu+\nu)}.$$

Therefore, when $k\lambda_{h,m} > 1$ we also have

$$\lambda_{h,m}^{\nu} \left| \left(R(k\lambda_{h,m})^j - e^{-\lambda_{h,m}kj} \right) \right|^2 \le Ct_j^{-(\mu+\nu)} k^{\mu}.$$

These inequalities together with Parseval's identity yield

$$T_1^2 \le Ct^{-(\mu+\nu)}k^{\mu} \sum_{m=1}^{\infty} \left(A_h^{-\frac{\nu}{2}} P_h x, \phi_{h,m} \right)^2 = Ct^{-(\mu+\nu)}k^{\mu} \left\| A_h^{-\frac{\nu}{2}} P_h x \right\|$$
$$\le Ct^{-(\mu+\nu)}k^{\mu} \|x\|_{-\nu}$$

when $\nu \in \{0,1\}$. The intermediate cases again follow by the same interpolation technique.

The term T_2 is the direct result of Lemma 4.3 (ii) as

$$T_2 = \|F_h(t_j)x\| \le Ct_j^{-\frac{\mu+\nu}{2}} h^{\mu} \|x\|_{-\nu} \le Ct^{-\frac{\mu+\nu}{2}} h^{\mu} \|x\|_{-\nu}.$$

For the last term T_3, we have

$$T_3 = \left\| A^{\frac{\mu+\nu}{2}} E(t) A^{-\frac{\mu}{2}} \left(E(t_j - t) - \mathrm{Id} \right) A^{-\frac{\nu}{2}} x \right\|.$$

With Lemma 2.5 (i) and (ii), the RHS gives

$$\left\| A^{\frac{\mu+\nu}{2}} E(t) A^{-\frac{\mu}{2}} \left(E(t_j - t) - \mathrm{Id} \right) A^{-\frac{\nu}{2}} x \right\| \le Ct^{-\frac{\mu+\nu}{2}} (t_j - t)^{\frac{\mu}{2}} \|x\|_{-\nu}$$
$$\le Ct^{-\frac{\mu+\nu}{2}} k^{\frac{\mu}{2}} \|x\|_{-\nu}.$$

Together the estimates for T_1, T_2 and T_3 complete the proof. □

Lemma 4.8. *Under Assumption 4.1, let* $0 \le \mu \le 1$, $0 \le \nu \le 1$ *and* $\mu + \nu < 1$, *there exists a constant* C *such that*

$$\left(\int_0^t \|F_{kh}(s)x\|^2 ds \right)^{\frac{1}{2}} \le C \left(h^{\mu} + k^{\frac{\mu}{2}} \right) \|x\|_{-\nu} \ \ \forall x \in \dot{H}^{-\nu},\ t > 0,\ \text{and}\, h,k \in (0,1].$$

Proof. With $\mu + \lambda < 1$ and Lemma 4.7 (ii) we have

$$\begin{aligned}
\left(\int_0^t \|F_{kh}(s)x\|^2 ds \right)^{\frac{1}{2}} &\leq \left(\int_0^t C(h^\mu + k^{\frac{\mu}{2}})^2 s^{-(\mu+\nu)} \|x\|^2_{-\nu} \right)^{\frac{1}{2}} \\
&\leq C \left((h^\mu + k^{\frac{\mu}{2}})^2 \|x\|^2_{-\nu} \frac{t^{1-(\mu+\nu)}}{1-(\mu+\nu)} \right)^{\frac{1}{2}} \\
&\leq C \left((h^\mu + k^{\frac{\mu}{2}})^2 \|x\|^2_{-\nu} T^{1-(\mu+\nu)} \right)^{\frac{1}{2}} \\
&\leq C \left(h^\mu + k^{\frac{\mu}{2}} \right) \|x\|_{-\nu}.
\end{aligned}$$

□

Let U_h^j denote the approximation of the mild solution U to (3.4) at time $t_j = jk$. A full discretization of (3.4) results from the backward Euler scheme as

$$\begin{aligned}
U_h^j - U_h^{j-1} + k \left[A_h U_h^j + P_h f(t, U_h^{j-1}) \right] dt &= P_h b(t, U_h^{j-1}) \Delta W_{H_1}^j \\
&\quad + P_h(\delta + A) N_\delta c(t, U_h^{j-1}) \Delta W_{H_2}^j, \\
U_h^0 &= P_h U_0,
\end{aligned} \tag{4.31}$$

where $\Delta W^j := W(t_j) - W(t_{j-1})$ denotes the increment of a cylindrical Wiener process.

Then we have the next theorem with respect to the estimate of $\|U_h^j - U(t_j)\|_{L^p(\Omega;H)}$:

Theorem 4.9. *Under the assumptions in Chapter 3, Assumption 4.1 and Assumption 4.5, for $r \in [0,1)$ and $p \in [2,\infty)$, there exists a constant C, independent of $h, k \in (0,1]$, such that*

$$\|U_h^j - U(t_j)\|_{L^p(\Omega;H)} \leq C \left(h^r + k^{\frac{r}{2}} \right), \quad \forall j = 1, 2, \dots, N_k,$$

where $U(t_j)$ is the mild solution to (3.4) and and U_h^j is given by (4.31).

Proof. With the solution operator $R(kA_h)$ defined in (4.26), the solution of

(4.31) is written as

$$
\begin{aligned}
U_h^j = R(kA_h)^j P_h U_0 - k\sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h f(t_i, U_h^i) \\
+ \sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h b(t_i, U_h^i)\Delta W_{H_1}^{i+1} \\
+ \sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h (\delta + A) N_\delta c(t_i, U_h^i)\Delta W_{H_2}^{i+1}.
\end{aligned}
$$

Hence the norm of the error is

$$
\begin{aligned}
&\|U_h^j - U(t_j)\|_{L^p(\Omega;H)} \\
&\quad \leq \left\| R(kA_h)^j P_h U_0 - E(t_j)U_0 \right\|_{L^p(\Omega;H)} \\
&\qquad + \left\| k\sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h f(t_i, U_h^i) - \int_0^{t_j} E(t_j - s) f(s, U(s)) ds \right\|_{L^p(\Omega;H)} \\
&\qquad + \left\| \sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h b(t_i, U_h^i)\Delta W_{H_1}^{i+1} \right. \\
&\qquad\quad \left. - \int_0^{t_j} E(t_j - s) b(s, U(s)) dW_{H_1}(s) \right\|_{L^p(\Omega;H)} \\
&\qquad + \left\| \sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h (\delta + A) N_\delta c(t_i, U_h^i)\Delta W_{H_2}^{i+1} \right. \\
&\qquad\quad \left. - \int_0^{t_j} E(t_j - s)(\delta + A) N_\delta c(s, U(s)) dW_{H_2}(s) \right\|_{L^p(\Omega;H)}
\end{aligned}
\tag{4.32}
$$

with $U(t_j)$ is given by (3.11).

The first term is a direct result of Lemma 4.7 (i) with $\mu = \nu = r$, such that

$$
\left\| R(kA_h)^j P_h U_0 - E(t_j)U_0 \right\|_{L^p(\Omega;H)} \leq C\left(h^r + k^{\frac{r}{2}}\right) \left\| A^{\frac{r}{2}} U_0 \right\|_{L^p(\Omega;H)},
$$

given $U_0(\omega) \in \dot{H}^r$ by Assumption 3.5.

For the rest terms we introduce three processes defined by

$$\tilde{f}(t) := f(t_{j-1}, U_h^{j-1}), \qquad \text{for } t \in (t_{j-1}, t_j], \quad j = 1, \ldots, N_k,$$
$$\tilde{f}(0) := f(t_0, U_0),$$

$$\tilde{b}(t) := b(t_{j-1}, U_h^{j-1}), \qquad \text{for } t \in (t_{j-1}, t_j], \quad j = 1, \ldots, N_k,$$
$$\tilde{b}(0) := b(t_0, U_0),$$

and

$$\tilde{c}(t) := (\delta + A)N_\delta c(t_{j-1}, U_h^{j-1}), \qquad \text{for } t \in (t_{j-1}, t_j], \quad j = 1, \ldots, N_k,$$
$$\tilde{c}(0) := (\delta + A)N_\delta c(t_0, U_0).$$

Hence with $E_{kh}(t), t \geq 0$ defined by (4.27), we have

$$\begin{aligned} k\sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h f(t_i, U_h^i) &= \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} E_{kh}(t_j - s) P_h \tilde{f}(s) ds \\ &= \int_0^{t_j} E_{kh}(t_j - s) P_h \tilde{f}(s) ds, \end{aligned} \tag{4.33}$$

thus similarly,

$$\sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h (\delta + A) b(t_i, U_h^i) \Delta W_{H_1}^{i+1} = \int_0^{t_j} E_{kh}(t_j - s) P_h \tilde{b}(s) dW_{H_1}(s), \tag{4.34}$$

and

$$\sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h (\delta + A) N_\delta c(t_i, U_h^i) \Delta W_{H_2}^{i+1} = \int_0^{t_j} E_{kh}(t_j - s) P_h \tilde{c}(s) dW_{H_2}(s). \tag{4.35}$$

By substituting the discrete term of the second summand in (4.32) for

(4.33), we obtain

$$
\begin{aligned}
&\left\| k\sum_{i=0}^{j-1} R(kA_h)^{j-i}P_h f(t_i, U_h^i) - \int_0^{t_j} E(t_j - s)f(s, U(s))ds \right\|_{L^p(\Omega;H)} \\
&= \left\| \int_0^{t_j} E_{kh}(t_j - s)P_h\tilde{f}(s)ds - \int_0^{t_j} E(t_j - s)f(s, U(s))ds \right\|_{L^p(\Omega;H)} \\
&\le \left\| \int_0^{t_j} E_{kh}(t_j - s)P_h\left(\tilde{f}(s) - f(s, U(s))\right)ds \right\|_{L^p(\Omega;H)} \\
&\quad + \left\| \int_0^{t_j} F_{kh}(t_j - s)f(s, U(s))ds \right\|_{L^p(\Omega;H)} \\
&=: J_1 + J_2.
\end{aligned}
$$

From (4.28) and Lemma 3.7, we have the first term J_1 as

$$
\begin{aligned}
J_1 &\le \int_0^{t_j} \left\| E_{kh}(t_j - s)P_h\left(\tilde{f}(s) - f(s, U(s))\right) \right\|_{L^p(\Omega;H)} ds \\
&\le C\sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} t_{j-i}^{-\frac{1}{2}} \left\| f(t_i, U_h^i) - f(s, U(s)) \right\|_{L^p(\Omega;\dot{H}^{-1})} ds \\
&\le Ck\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\| f(t_i, U_h^i) - f(t_i, U(t_i)) \right\|_{L^p(\Omega;\dot{H}^{-1})} \\
&\quad + C\sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} t_{j-i}^{-\frac{1}{2}} \left\| f(t_i, U(t_i)i) - f(s, U(s)) \right\|_{L^p(\Omega;\dot{H}^{-1})} ds \\
&\le Ck\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\| U_h^i - U(t_i) \right\|_{L^p(\Omega;H)} \\
&\quad + C\left(1 + \sup_{t\in[0,T]} \|U(t)\|_{L^p(\Omega;H)}\right) \sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \int_{t_i}^{t_{i+1}} (s - t_i)^{\frac{r}{2}} ds,
\end{aligned}
$$

where

$$\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \int_{t_i}^{t_{i+1}} (s-t_i)^{\frac{r}{2}} ds = \frac{1}{1+\frac{r}{2}} k^{1+\frac{r}{2}} \sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \leq C k^{\frac{r}{2}} \int_0^{t_j} s^{-\frac{1}{2}} ds \leq C T^{\frac{1}{2}} k^{\frac{r}{2}}.$$

Thus the term J_1 is estimated by

$$J_1 \leq C k^{\frac{r}{2}} + C k \sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\| U_h^i - U(t_i) \right\|_{L^p(\Omega;H)}. \tag{4.36}$$

For the term J_2, we apply Lemma 4.7 (ii) with $\mu = r$ and $\nu = -1$, Lemma 3.7 and the similar argument giving (4.15) to obtain

$$J_2 \leq C \left(h^r + k^{\frac{r}{2}} \right) \tag{4.37}$$

Next the term of $b(t_i, U_h^i)$ uses the substitution of (4.34), which produces

$$\begin{aligned}
&\left\| \sum_{i=0}^{j-1} R(kA_h)^{j-i} P_h b(t_i, U_h^i) \Delta W_{H_1}^{i+1} - \int_0^{t_j} E(t_j - s) b(s, U(s)) dW_{H_1}(s) \right\|_{L^p(\Omega;H)} \\
&= \left\| \int_0^{t_j} E_{kh}(t_j - s) P_h \tilde{b}(s) dW_{H_1}(s) - \int_0^{t_j} E(t_j - s) b(s, U(s)) dW_{H_1}(s) \right\|_{L^p(\Omega;H)} \\
&\leq C \left(\mathbf{E} \left[\left(\int_0^{t_j} \left\| E_{kh}(t_j - s) P_h \tilde{b}(s) - E(t_j - s) b(s, U(s)) \right\|_{L_2}^2 ds \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}.
\end{aligned}$$

Then the triangle inequality of stochastic integral implies

$$
\begin{aligned}
&\left(\mathbf{E}\left[\left(\int_0^{t_j} \left\|E_{kh}(t_j-s)P_h\tilde{b}(s)-E(t_j-s)b(s,U(s))\right\|_{L_2}^2 ds\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}\\
&\le \left\|\left(\int_0^{t_j} \left\|E_{kh}(t_j-s)P_h\left(\tilde{b}(s)-b(s,U(s))\right)\right\|_{L_2}^2 ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}\\
&\quad+\left\|\left(\int_0^{t_j} \left\|F_{kh}(t_j-s)\left(b(s,U(s))-b(t_j,U(t_j))\right)\right\|_{L_2}^2 ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})} \qquad (4.38)\\
&\quad+\left\|\left(\int_0^{t_j} \left\|F_{kh}(t_j-s)b(t_j,U(t_j))\right\|_{L_2}^2 ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}\\
&=: J_3+J_4+J_5.
\end{aligned}
$$

The term J_3 is estimated by the smoothing property (4.4), the fact $\|P_hx\|\le\|x\|$ for all $x\in H$ and Lemma 3.7 as follows

$$
\begin{aligned}
J_3 &\le C\left\|\left(\int_0^{t_j} \left\|\tilde{b}(s)-b(s,U(s))\right\|_{L_2}^2 ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}\\
&\le C\left(\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}} \left\|b(t_i,U_h^i)-b(s,U(s))\right\|_{L^p(\Omega;H)}^2 ds\right)^{\frac{1}{2}}\\
&\le C\left(k\sum_{i=0}^{j-1} \left\|b(t_i,U_h^i)-b(t_i,U(t_i))\right\|_{L^p(\Omega;H)}^2 ds\right)^{\frac{1}{2}}\\
&\quad+C\left(\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}} \left\|b(t_i,U(t_i))-b(s,U(s))\right\|_{L^p(\Omega;H)}^2 ds\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq C\left(k\sum_{i=0}^{j-1}\left\|U_h^i-U(t_i)\right\|^2_{L^p(\Omega;H)}\right)^{\frac{1}{2}}$$
$$+C\left(1+\sup_{t\in[0,T]}\|U(t)\|_{L^p(\Omega;H)}\right)\left(\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(s-t_i)^r ds\right)^{\frac{1}{2}}$$
$$\leq CT^{\frac{1}{2}}k^{\frac{r}{2}}+C\left(k\sum_{i=0}^{j-1}\left\|U_h^i-U(t_i)\right\|^2_{L^p(\Omega;H)}\right)^{\frac{1}{2}} \tag{4.39}$$

In the same way of estimating (4.18), by Lemma 4.7 (ii) with $\mu=r$ and $\nu=0$ we have the estimate of J_4 as

$$J_4\leq C\left(h^r+k^{\frac{r}{2}}\right). \tag{4.40}$$

Similarly the estimate of the term J_5 is derived by Lemma 4.8 with $\mu=r,\nu=0$ and the same arguments which gave (4.19) as follows

$$J_5\leq C\left(h^r+k^{\frac{r}{2}}\right). \tag{4.41}$$

Finally we apply the substitution of (4.35) and the inequality in (4.38) on the last summand in (4.32) to obtain

$$\left\|\sum_{i=0}^{j-1}R(kA_h)^{j-i}P_h(\delta+A)N_\delta c(t_i,U_h^i)\Delta W_{H_2}^{i+1}\right.$$
$$\left.-\int_0^{t_j}E(t_j-s)(\delta+A)N_\delta c(s,U(s))dW_{H_2}(s)\right\|_{L^p(\Omega;H)}$$
$$=\left\|\int_0^{t_j}E_{kh}(t_j-s)P_h\tilde{c}(s)dW_{H_2}(s)\right.$$
$$\left.-\int_0^{t_j}E(t_j-s)(\delta+A)N_\delta c(s,U(s))dW_{H_2}(s)\right\|_{L^p(\Omega;H)}$$

$$
\begin{aligned}
&\leq C\left(\mathbf{E}\left[\left(\int_0^{t_j}\left\|E_{kh}(t_j-s)P_h\tilde{c}(s)-E(t_j-s)(\delta+A)N_\delta c(s,U(s))\right\|_{L_2}^2 ds\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}\\
&\leq \left\|\left(\int_0^{t_j}\left\|E_{kh}(t_j-s)P_h\left(\tilde{c}(s)-(\delta+A)N_\delta c(s,U(s))\right)\right\|_{L_2}^2 ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}\\
&\quad+\left\|\left(\int_0^{t_j}\left\|F_{kh}(t_j-s)(\delta+A)N_\delta\left(c(s,U(s))-c(t_j,U(t_j))\right)\right\|_{L_2}^2 ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}\\
&\quad+\left\|\left(\int_0^{t_j}\left\|F_{kh}(t_j-s)(\delta+A)N_\delta c(t_j,U(t_j))\right\|_{L_2}^2 ds\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}\\
&=: J_6+J_7+J_8.
\end{aligned}
$$

For the term J_6 and J_7, they are analogous to (4.39) and (4.40). Hence they have the estimate

$$
J_6 \leq Ck^{\frac{r}{2}} + C\left(k\sum_{i=0}^{j-1}\left\|U_h^i-U(t_i)\right\|_{L^p(\Omega;H)}^2\right)^{\frac{1}{2}}, \tag{4.42}
$$

and the estimate

$$
J_7 \leq C\left(h^r+k^{\frac{r}{2}}\right), \tag{4.43}
$$

respectively.

The estimate of the term J_8 follows the same arguement of deriving (4.23) and also Lemma 4.8 such that we have

$$
J_8 \leq C\left(h^r+k^{\frac{r}{2}}\right). \tag{4.44}
$$

Altogether by (4.36)-(4.37), (4.39)-(4.41) and (4.42)-(4.44), we have the estimate of (4.32) given by

$$\begin{aligned}\|U_h^j - U(t_j)\|^2_{L^p(\Omega;H)} \leq C\left(h^r + k^{\frac{r}{2}}\right)^2 \\ + C\left(k\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\|U_h^i - U(t_i)\right\|_{L^p(\Omega;H)}\right)^2 \\ + Ck\sum_{i=0}^{j-1} \left\|U_h^i - U(t_i)\right\|^2_{L^p(\Omega;H)}.\end{aligned}$$

Also we have

$$k\sum_{i=0}^{j-1} \left\|U_h^i - U(t_i)\right\|^2_{L^p(\Omega;H)} \leq T^{\frac{1}{2}} k\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\|U_h^i - U(t_i)\right\|^2_{L^p(\Omega;H)}$$

and by Hölder's inequality

$$\begin{aligned}&k\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\|U_h^i - U(t_i)\right\|^2_{L^p(\Omega;H)} \\ &\leq \left(k\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}}\right)^{\frac{1}{2}} \left(k\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\|U_h^i - U(t_i)\right\|^2_{L^p(\Omega;H)}\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^{t_j} s^{-\frac{1}{2}} ds\right)^{\frac{1}{2}} \left(k\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\|U_h^i - U(t_i)\right\|^2_{L^p(\Omega;H)}\right)^{\frac{1}{2}} \\ &\leq \left(2T^{\frac{1}{2}}\right)^{\frac{1}{2}} \left(k\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\|U_h^i - U(t_i)\right\|^2_{L^p(\Omega;H)}\right)^{\frac{1}{2}}.\end{aligned}$$

Hence we have proved that

$$\|U_h^j - U(t_j)\|^2_{L^p(\Omega;H)} \leq C\left(h^r + k^{\frac{r}{2}}\right)^2 + Ck\sum_{i=0}^{j-1} t_{j-i}^{-\frac{1}{2}} \left\|U_h^i - U(t_i)\right\|^2_{L^p(\Omega;H)}$$

and by applying Lemma A.1 we complete the proof. □

4.4 Numerical experiments

In this section we show some numerical experiments and results. As a start, we identify and discretize the noise. Though noise discretization is not the focus of the thesis as stated in **Chapter 1**, it helps in justifying and explaining the results of the numerical experiments. Recall that for the SPDEs driven by additive noise, it is classic that the H-valued Wiener process admits a Karhunen-Loève expansion

$$W(t) = \sum_{m=1}^{\infty} \lambda_m^{\frac{1}{2}} \beta_m(t) e_m,$$

where $\beta_m(t)$ are independent real value Brownian motions, (λ_m) and (e_m) are eigenvalues and eigenvectors of the covariance operator Q. The fact that

$$\mathbf{E}\|W(t)\|^2 = \sum_{m=1}^{\infty} \lambda_m \mathbf{E}(\beta_m(t))^2 = t\mathrm{Tr}(Q) < \infty,$$

requires that the trace of Q denoted by $\mathrm{Tr}(Q) < \infty$. Hence if $Q = \mathrm{Id}$, i.e., $\lambda_m = 1$, then $W(t)$ is not H-valued. However, in the practice with Galerkin FEM, the expansion of $W(t)$ can only be truncated to finite

$$W(t) = \sum_{m=1}^{N_h} \lambda_m^{\frac{1}{2}} \beta_m(t) e_m,$$

where N_h denotes the dimension of a finite dimensional subspace $S_h \subset \dot{H}^1$ introduced at the beginning of the chapter. Thus $\mathbf{E}\|W(t)\|^2 < \infty$ is always satisfied and irrelevant to the choice of (λ_m). Consequently the eigenvalues and eigenvectors can be conveniently chosen as $\lambda_m = 1$ and $e_m = \sqrt{2}\sin(m\pi x)$ for $x \in D$, such that (e_m) is an orthonormal basis of S_h if Galerkin spectral method is considered in space discretization (See [16], [28] and [50]). Moreover, by truncating to finite dimension, the regularity of $W(t)$ in space is increased significantly to the extent that $W(t)$ can be seen as a C^∞-valued noise with large variance.

It is shown clearly in Chapter 2 of [30] that by assuming more regular initial condition U_0, forcing term $f(t, U(t))$ and driving noise $b(t, U(t))dW_{H_1}(t)$, the solution of the SPDE driven by additive noise with homogeneous boundary conditions exists in $\dot{H}^{1+r}$ with $r \in [0, 1)$ and consequently the optimal convergence rate shown in Chapter 3 is h^{1+r} in space and $k^{\frac{1}{2}}$ in time, where h is the mesh size and k is the time stepping size.

If the best proxy of "true" solution is the approximation of it when N_h is very large, then the experimental "true" solution is more regular than the theoretical one. Hence we expect that the experimental convergence rate is better than the theoretical one. The numerical experiment in [50] makes this point very obvious.

The same argument applies for the SPDEs with Neumann boundary noise and the numerical results that we now present exhibit the same phenomena. Similar to Chapter 15 of [34], we introduce a heat equation with Neumann boundary noise on a unit ball $D \in \mathbb{R}^n$ given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\Delta u, \ t \in (0,1], x \in D \\ u(0,x) &= 100\sin(\pi\|x\|), \\ \frac{\partial u}{\partial \vec{n}}(t,x) &= \dot{W}(t,x), \ x \in \partial D \end{aligned} \tag{4.45}$$

We assume the Wiener process $W(t)$ has the form

$$W(t) = \sum_{m=1}^{\infty} \gamma_m \beta_m(t) \varphi_m, \tag{4.46}$$

where β_m are independent Brownian motions, (γ_m) is a sequence of real numbers and (φ_m) is a sequence of functions defined on the boundary ∂D.

By [34, Corollary 8.7], the stochastic integral in (3.11)

$$\int_0^t E(t-s)(\delta + A)N_\delta c(t, U(t))dW_{H_2}(t),$$

is well-defined if and only if

$$\int_0^t \|E(t-s)(\delta + A)N_\delta\|_{L_2} ds < \infty, \tag{4.47}$$

given Assumption 3.4. Hence for the heat equation (4.45) with the Wiener process given by (4.46), the condition (4.47) is equivalent to

$$\sum_{m=1}^{\infty} \gamma_m^2 \|N_\delta \varphi_m\|_1^2 < \infty.$$

Hence (φ_m) can well be the trace of functions in $H^1(D)$, which exists in the sense of Proposition 2.8, such that $W(t)$ is $L^2(\partial D)$-valued. Another obvious choice of (φ_m) is the restriction of functions in $C^\infty(D)$ on the manifold of $\partial D \in \mathbb{R}^{n-1}$.

In the numerical computation, the expansion of (4.46) can only be truncated to the finite dimension, i.e.,

$$W(t) = \sum_{m=1}^{B_h} \gamma_m \beta_m(t) \varphi_m, \tag{4.48}$$

where B_h is the number of "edges" on the sphere ∂D. With the truncation, again the value of γ_m is irrelevant in satisfying the condition (4.47) and we have the liberty to choose very smooth φ_m, e.g., $\varphi_m = \cos(m\pi\|x\|)$ for $x \in \partial D$, which is naturally extended from the boundary to the domain.

Thus, the choice of γ_m only decides the "size" of perturbation on the boundary but not change the fact that the equation (4.45) is downgraded to an equation with perturbation on the boundary for each realization or simulation of the noise. The existence and regularity of solution to a PDE with such perturbation was discussed in Chapter 4 [19]. Equations that are perturbed periodically on the boundary were specially considered in [46]. Moreover, if the boundary condition has finite numbers of discontinuity or jumps, it was proved in [52] that the discontinuity will not be propagated into the space-time domain. In this case, it was discussed extensively in the literature above that the solution of the PDE exists in $\dot{H}^1$ and it is well known that the convergence rate of Galerkin FEM is optimal, i.e., h^1 in space and $k^{\frac{1}{2}}$ in time. This coincides with the claim made at the beginning of the thesis, that the numerical experiments help to verify the theoretical results but does not show the full infinite dimensional behaviour exhibited int the theory.

Now we show the experiment results of (4.45) when the dimension is $d = 1$ with the domain $D = (0, 1)$. Then the boundary $\partial D = \{0, 1\}$. Therefore the Wiener process $W(t)$ on $L^2(\partial D)$ is identified as $W(t) = (w_1(t), w_2(t))$ where $w_1(t)$ and $w_2(t)$ are real-valued Brownian motions. To see how the noise on the boundary affects the solution compared to the deterministic case, we set $w_1(t) = 0$ such that the boundary condition is

$$\frac{\partial u}{\partial \vec{n}}(t, 0) = 0, \ \frac{\partial u}{\partial \vec{n}} = \dot{w}(t) \quad \text{for } t \in (0, 1],$$

where $w(t)$ is a real-valued Brownian motion.

In the experiment, first we see the distribution of the simulations of the solution to (4.45)[3]. The mean $\mathbf{E}(u(t, x))$ and the average of simulated solutions are shown in 4.1.

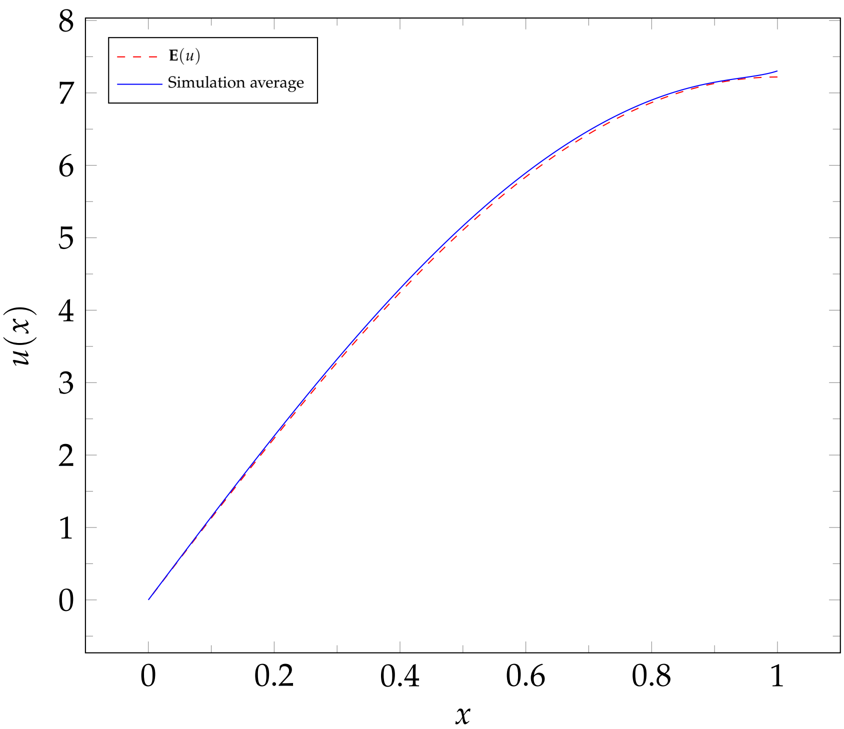


Figure 4.1: Average of simulations and $\mathbf{E}(u)$

The mean $\mathbf{E}(u(t, x))$ in Figure 4.1 is the solution of the deterministic version of (4.45) with the homogeneous boundary condition as a result of the Leibniz integral rule. We can only find a proxy of "true" solutions to (4.45) and its deterministic counterpart by a numerical approximation. By Theorem 4.9, the approximation is getting closer to the "true" solution when h and k are sufficiently small. Hence we compute the solutions with small time step $k = 2^{-10}$ and space step $h = 2^{-8}$ as the "true" solutions. We consider $M = 100$

[3]The computation is implemented with FEniCS [42] throughout the thesis.

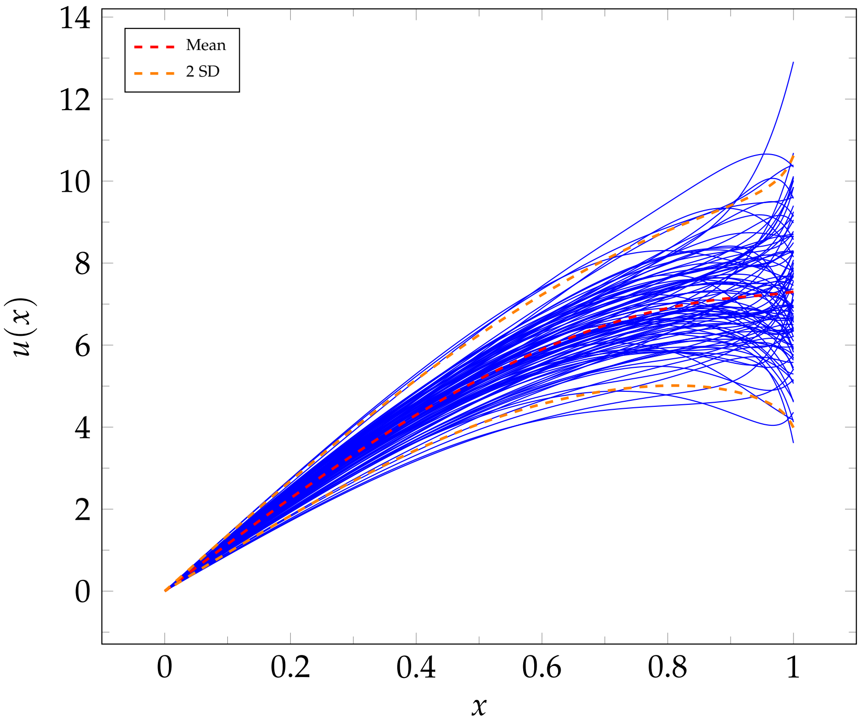


Figure 4.2: Distribution of simulations of solution

simulations as it is large enough to observe the empirical distribution of simulated solutions. As we can see, the average of simulations converges to the mean. Since it is not easy to compute the second moment $\mathbf{E}(u(t,x))^2$ in a direct manner, we approximate it by computing the sample moment with

$$\mathbf{E}(u(t,x))^2 \approx \frac{1}{M}\sum_{l=1}^{M}(u_l(t,x)^2),$$

where $u_l(t,x)$ is the solution to l-th simulation. We show 2 standard deviation of the simulated data in Figure 4.2.

Next we intend to see if the errors computed with the scheme depend on the regularity order r as stated in Theorem 4.9. First we fix the time step $k = 2^{-10}$ and for each simulation $\omega_l, l = 1, \dots, M$, we compute the numerical approximation $U^n_{h_i}(\omega_l)$ at the terminal time $t_n = 1$ with coarser mesh size $h_i = 2^{-i}, i = 1, \dots, 5$ than the one used in computing the "true" solution. Then we compute the following L^2-norm of the error

$$\epsilon(h_i, \omega_l) = \left\| U^n_{h_i}(\omega_l) - u(t_n, \omega_l) \right\|^2.$$

The L^2-norm $\|U^n_{h_i} - u(t_n)\|_{L^2(\Omega;H)}$ is approximated by the average over all simulations ω_l

$$\mathcal{E}(h_i, k) = \left(\frac{1}{M} \sum_{l=1}^{M} \epsilon(h_i, \omega_l) \right)^{\frac{1}{2}}.$$

Since $r \in [0, 1)$, the convergence order is almost $\mathcal{O}(h^1)$. Hence we expect that

$$\frac{\mathcal{E}(h_i, k)}{\mathcal{E}(h_{i+1}, k)} \approx \left(\frac{h_i}{h_{i+1}} \right) = 2$$

for the sufficiently small time step $k = 2^{-10}$.

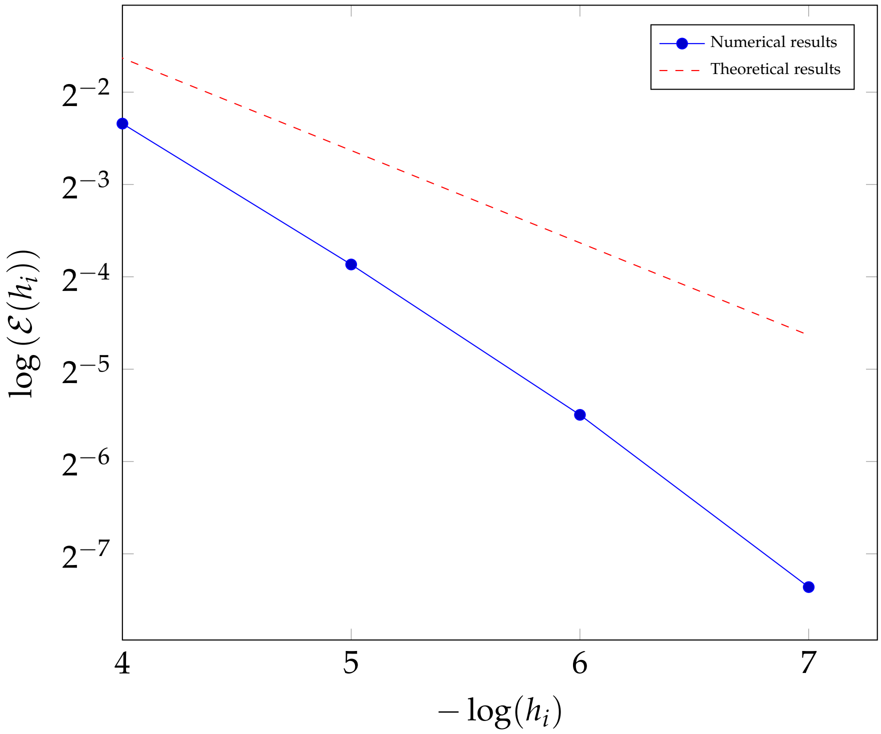
The results of experiment are presented in Figure 4.3a and Table 4.3b. With the mesh size being finer, the error in terms of L^2-norm is smaller. The ratio also coincides with the expectation. The reason that the convergence order of the experiment appears higher than the estimate is due to very regular initial condition and the smoothness of the boundary noise by truncating (see (4.48)).

In a similar way, we fix the mesh size $h = 2^{-8}$ and compute the L^2-norm of the error with different time step $k_i = 2^{-i}, i = 6, \ldots, 9$, which are bigger than the one used in approximating the "true" solution. The computation results in Figure 4.4a and Table 4.4b show that the convergence order coincides with the result of Theorem 4.9, i.e.,

$$\frac{\mathcal{E}(k_i, h)}{\mathcal{E}(k_{i+1}, h)} \approx \left(\frac{k_i}{k_{i+1}} \right) = 2^{\frac{1}{2}} \approx 1.41.$$

We also present the table of the approximation of $\mathbf{E}u(t, x)$ and $\mathbf{E}(u(t, x))^2$ at $t = 1, x = 0.5$ with different time step and mesh size in Table 4.1. It shows that the computational results converge when $h, k \to 0$, which is consistent with the analysis above.

(a) L^2-norm errors in different mesh size

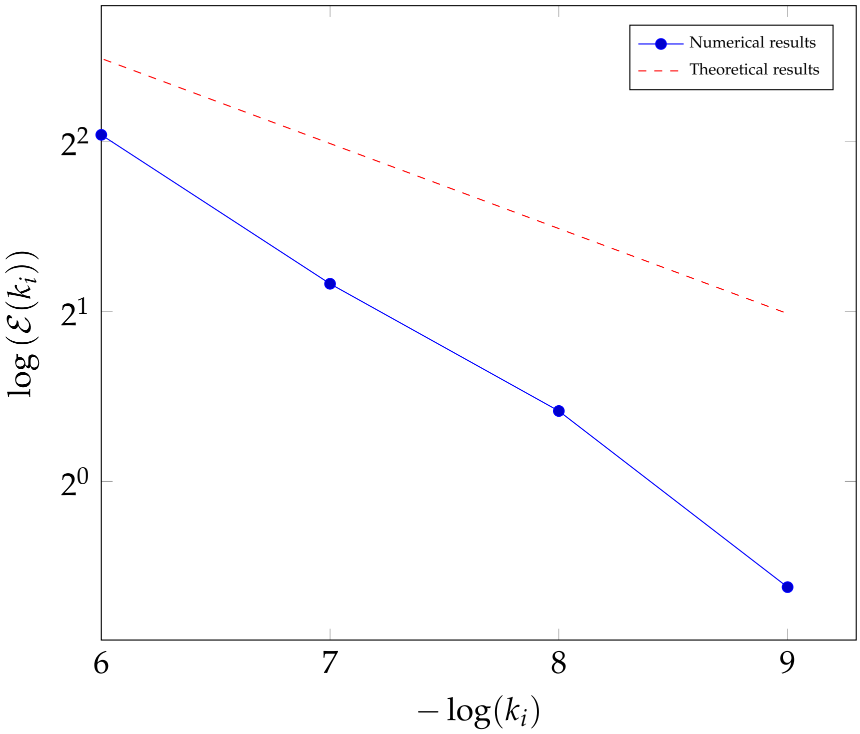


(b) Ratios of errors with different mesh size

i	h_i	$\mathcal{E}(h_i)$	$\mathcal{E}(h_i)/\mathcal{E}(h_{i+1})$
4	2^{-4}	0.1975	2.88
5	2^{-5}	0.0686	3.09
6	2^{-6}	0.0222	3.64
7	2^{-7}	0.0061	

Figure 4.3: Errors in L^2-norm with time stepping size $k = 2^{-10}$

(a) L^2-norm erros in different time step



(b) Ratios of errors with different time step

i	k_i	$\mathcal{E}(k_i)$	$\mathcal{E}(k_i)/\mathcal{E}(k_{i+1})$
6	2^{-6}	4.1044	1.83
7	2^{-7}	2.2370	1.68
8	2^{-8}	1.3319	2.05
9	2^{-9}	0.6498	

Figure 4.4: Errors in L^2-norm with mesh size $h = 2^{-8}$

Table 4.1: $\mathbf{E}(u)$ and $\mathbf{E}(u^2)$ at $t = 1, x = 0.5$

h	k	$\mathbf{E}(u(1, 0.5))$	$\mathbf{E}(u(1, 0.5))^2$
1/4	1/4	8.3480	69.9013
1/4	1/8	6.6170	44.0246
1/4	1/16	5.7197	32.9741
1/4	1/32	5.2595	27.9412
1/8	1/8	6.9415	48.4240
1/8	1/16	6.0135	36.4204
1/8	1/32	5.5380	30.9447
1/16	1/16	6.0876	37.3170
1/16	1/32	5.6083	31.7279
1/256	1/1024	5.1627	26.9481

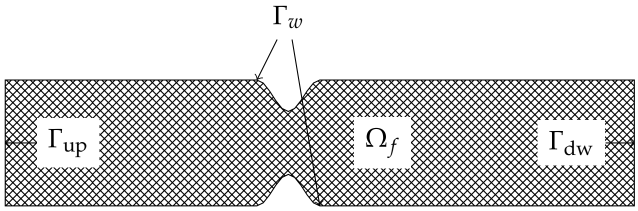
Chapter 5

Modeling solute dynamics in the vascular system with SPDE

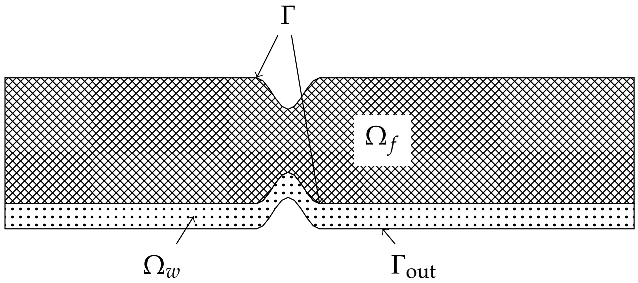
In this chapter, we apply our results on numerical approximation of SPDE with boundary noise to the modeling of solute dynamics in the vascular system. This application is inspired by the paper [37] which studied the blood solute dynamic in a two dimensional domain. A PDE system was proposed in modeling the blood solutes in the arteries. Two models were considered in the paper such that the blood solute concentration was modeled by an advection-diffusion equation and a coupled system of advection-diffusion equation and diffusion equation respectively. In both models, the blood solute concentration was determined by the blood flow. Hence a Navier-Stokes equation is used to model the velocity of blood flow in the artery. In their study, the boundary conditions of both models depended on the exchange of the blood solute over the artery walls (part of the boundary), and they posed them as functions of the wall permeability parameter ζ, where a linear regression was used to estimate ζ. We show that a noise is introduced naturally when modeling ζ such that it is time-varying and stochastic. Consequently the PDEs modeling the blood solute concentrate are extended to be equations with Neumann boundary noise.

5.1 The wall-free and fluid-wall models

In [37], the *the wall-free model* and *the fluid-wall model* are proposed to model the blood solutes dynamics as shown in Figure 5.1,



(a) The wall-free model: $\Omega_f \subset R^n$ is the lumen of a given vascular district with infinitesimally thin boundary Γ_w.



(b) The fluid-wall model: dynamics in the arterial wall Ω_w are included. Solute concentration at the interface Γ becomes an unknown.

Figure 5.1: Description of two models

In the wall-free model (Figure 5.1a), the vessel wall is very thin such that the solutes dynamics are negligible inside the wall. Hence the concentration of the solute outside the arterial walls denoted by κ_w is assigned as a known quantity. The concentration of the solute in the blood denoted by C_f is the unknown of the advection-diffusion equation given by

$$\begin{aligned}
&\frac{\partial C_f}{\partial t} - \nabla \cdot (\boldsymbol{\mu}_f \nabla C_f) + \mathbf{u} \nabla C_f = f_f, \quad \mathbf{x} \in \Omega_f,\ t \in (0, T],\\
&\mathbf{n} \cdot (\boldsymbol{\mu}_f \nabla C_f) + \zeta C_f = \zeta \kappa_w, \quad \mathbf{x} \in \Gamma_w,\ t \in (0, T],\\
&C_f = 0 \quad \text{on } \partial\Omega_f \backslash \Gamma_w,\ t \in (0, T],\\
&C_f = C_{f,0}, \quad \mathbf{x} \in \Omega_f,\ t = 0,
\end{aligned} \tag{5.1}$$

where ζ denotes the wall permeability and $\mathbf{u}$ is the blood flow velocity.

The diffusive tensor $\boldsymbol{\mu}_f$ in (5.1) is a function of the shear rate $\mathbf{d}$ given by

$$d_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, \tag{5.2}$$

with $(u_i)_{i=1,2}$ being the components of $\mathbf{u}$. Then a general model of $\boldsymbol{\mu}_f$ is proposed as follows

$$(\mu_f)_{ij} = \mu_{f0}(\delta_{ij} + \beta|d_{ij}|^\gamma), \quad i,j = 1,2, \tag{5.3}$$

where δ_{ij} is Kronecker delta. The choice the positive coefficients μ_{f0}, β and γ depends on the specific blood solute and the red blood cells concentration.

For the fluid-wall model (Figure 5.1b), because of the thickness of the vessel wall, the solutes dynamic in Ω_w has to be considered besides that in the vessel Ω_f. Hence κ_w on Γ_w is substituted by an unknown quantity C_w in the domain Ω_w. The value of C_w is given by a pure diffusion equation since the velocity of the blood is very low inside the wall. Thus the following equation systems arise from the fluid-wall model

$$\begin{aligned} &\frac{\partial C_f}{\partial t} - \nabla \cdot (\boldsymbol{\mu}_f \nabla C_f) + \mathbf{u} \cdot \nabla C_f = f_f \quad \text{in } \Omega_f,\ t \in (0,T], \\ &C_f = 0 \quad \text{on } \partial\Omega_f \backslash \Gamma,\ t \in (0,T], \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\frac{\partial C_w}{\partial t} - \nabla \cdot (\mu_w \nabla C_w) = f_w \quad \text{in } \Omega_w,\ t \in (0,T], \\ &C_w = 0 \quad \text{on } \partial\Omega_w \backslash \Gamma,\ t \in (0,T]. \end{aligned} \tag{5.5}$$

On the sharing boundary Γ, the coupled boundary conditions were

$$\begin{aligned} &\mathbf{n}_w \cdot (\mu_w \nabla C_w) + \zeta(C_f - C_w) = 0 \ \text{ on } \Gamma, \\ &\mathbf{n}_f \cdot (\boldsymbol{\mu}_f \nabla C_f) + \zeta(C_f - C_w) = 0 \ \text{ on } \Gamma, \end{aligned} \tag{5.6}$$

where $\mathbf{n}_f$ is the unit outward vector on Γ with respect to Ω_f and $\mathbf{n}_w = -\mathbf{n}_f$. The diffusive tensor $\boldsymbol{\mu}_f$ is defined as in the wall-free model and μ_w is a constant.

In both models, the solutes dynamics are related to the blood motion $\mathbf{u}$. The solutes are convected by the blood flow and absorbed through the arterial walls. The absorption results from the stress on the vascular tissue induced by the blood. The blood is assumed to be a Newtonian fluid with a constant viscosity ν such that the blood motion is governed by the Navier-Stokes equations given by

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla P = \mathbf{f}, \quad \mathbf{x}\in\Omega_f,\ t\in(0,T],\\
&\nabla\cdot\mathbf{u} = 0, \quad \mathbf{x}\in\Omega_f,\ t\in(0,T],\\
&\mathbf{u} = \mathbf{b} \quad \text{on } \partial\Omega_f\backslash\Gamma_w, \quad \mathbf{u} = 0 \text{ on } \Gamma_w, \quad t\in(0,T],\\
&\mathbf{u} = \mathbf{u}_0 \text{ and } \nabla\cdot\mathbf{u}_0 = 0, \quad \mathbf{x}\in\Omega_f,\ t\in(0,T],
\end{aligned} \tag{5.7}$$

where P denotes the kinematic pressure.

5.2 Formulation as SPDEs

The solute dynamic depends on the permeability of the vessel walls ζ. Hence one crucial issue for problem (5.1) and (5.4)-(5.5) is to choose the boundary value ζ. In [33], the authors demonstrated how to quantify the permeability ζ of the solute low-density lipoprotein with respect to the shear stress $\boldsymbol{\sigma}(\mathbf{u})$ within the frame of the fluid-wall model. For different type of solutes, the permeability of the solute is determined by the shear stress differently.

We model the concentration of oxygen the same as the test case in [37]. A linear relation is proposed between ζ and the shear pressure $\boldsymbol{\sigma}(\mathbf{u})$ for the oxygen given by

$$\zeta = K_1 + K_2|\boldsymbol{\sigma}(\mathbf{u})|.$$

The shear pressure is given by $\boldsymbol{\sigma}(\mathbf{u}) = \boldsymbol{\tau}\cdot\mathbf{T}\cdot\mathbf{n}$ on the boundary Γ, where $\boldsymbol{\tau}$ and $\mathbf{n}$ are the tangential and normal unit vectors on Γ respectively. The local stress tensor $\mathbf{T} = 2\nu\mathbf{d}$ with $\mathbf{d}$ being the shear rate given by (5.2).

The coefficients K_1 and K_2 are calculated with the mean oxygen flow over the vessel walls Γ:

$$\phi_{\text{mean}} = \frac{1}{\text{measure}(\Gamma)}\int_\Gamma \zeta\cdot(C_f - q_w)d\Gamma,$$

where q_w denotes the solute concentration in the wall, i.e., $q_w = \kappa_w$ for the wall-free model and $q_w = C_w$ for the fluid-wall model. We approximate the integral on the RHS by

$$\frac{1}{\text{measure}(\Gamma)}\int_\Gamma (C_f - q_w) \approx 0.3C_0,$$

where $C_0 = 2.58 \cdot 10^{-3}$ is the mean concentration of the oxygen in the lumen. Hence we obtain

$$0.3C_0 \cdot (K_1 + K_2|\boldsymbol{\sigma}(\mathbf{u})|) = \phi_{\text{mean}}.$$

By approximating $\boldsymbol{\sigma}(\mathbf{u})$ with $\boldsymbol{\sigma}(\mathbf{u})_0 = 1.98$ and assuming $K_1 = \frac{K_2\boldsymbol{\sigma}(\mathbf{u})_0}{2}$, we have $0.3C_0 \cdot \frac{3}{2}K_2\boldsymbol{\sigma}(\mathbf{u})_0 = \phi_{\text{mean}}$. Finally by taking $\phi_{\text{mean}} = 4.82 \cdot 10^{-6}$ as a reference value in [37], we have $K_1 = 2.09 \cdot 10^{-3}$ and $K_2 = 2.11 \cdot 10^{-3}$.

In practice, both the flux ϕ_{mean} and the shear stress $\boldsymbol{\sigma}(\mathbf{u})_0$ can be perturbed by the noises. Hence, the permeability ζ can be modeled as a linear regression with respect to $\boldsymbol{\sigma}(\mathbf{u})$ as

$$\zeta = K_1 + K_2|\boldsymbol{\sigma}(\mathbf{u})| + \xi,$$

where ξ is a space-time noise. Then the permeability parameter ζ in the boundary terms in (5.1) and (5.6) is substituted with the value above. As a result, to model the solute dynamics in the arteries, we have the following equations for the wall-free model

$$\begin{aligned} &\frac{\partial C_f}{\partial t} - \nabla \cdot (\boldsymbol{\mu}_f \nabla C_f) + \mathbf{u}\nabla C_f = f_f, \quad \mathbf{x} \in \Omega_f,\ t \in (0, T], \\ &\mathbf{n} \cdot (\boldsymbol{\mu}_f \nabla C_f) = \tilde{\zeta} + \dot{W}(t) \quad \mathbf{x} \in \Gamma_w,\ t \in (0, T], \\ &C_f = 0 \quad \text{on } \partial\Omega_f \backslash \Gamma_w,\ t \in (0, T], \\ &C_f = C_{f,0}, \quad \mathbf{x} \in \Omega_f,\ t = 0, \end{aligned} \tag{5.8}$$

and for the fluid-wall model

$$\begin{aligned} &\frac{\partial C_f}{\partial t} - \nabla \cdot (\boldsymbol{\mu}_f \nabla C_f) + \mathbf{u} \cdot \nabla C_f = f_f \quad \text{in } \Omega_f,\ t \in (0, T], \\ &C_f = 0 \quad \text{on } \partial\Omega_f \backslash \Gamma,\ t \in (0, T], \\ &\mathbf{n}_f \cdot (\boldsymbol{\mu}_f \nabla C_f) = \tilde{\zeta} + \dot{W}(t) \ \text{ on } \Gamma, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} &\frac{\partial C_w}{\partial t} - \nabla \cdot (\mu_w \nabla C_w) = f_w \quad \text{in } \Omega_w,\ t \in (0, T], \\ &C_w = 0 \quad \text{on } \partial\Omega_w \backslash \Gamma,\ t \in (0, T]. \\ &\mathbf{n}_w \cdot (\mu_w \nabla C_w) = \tilde{\zeta} + \dot{W}(t) \ \text{ on } \Gamma, \end{aligned} \tag{5.10}$$

where **u** is the solution to the Navier-Stokes equation (5.7), and

$$\tilde{\zeta} = \mathcal{M}(q_w - C_f)(K_1 + K_2|\boldsymbol{\sigma}(\mathbf{u})|), \; \dot{W}(t) = \mathcal{M}(q_w - C_f)\xi, \tag{5.11}$$

with $q_w = \kappa_w$ for the wall-free model and $q_w = C_w$ for the fluid-wall model, and $\mathcal{M}$ being linear mapping, which we discuss in details in the next section.

5.3 Solutions of the SPDEs

As the velocity field **u** presents in the coefficients and boundary conditions in (5.8)-(5.11), the well-poseness of the problems depends on the regularity of **u**. The existence and uniqueness of solutions to Navier-Stokes equations has been proved in many literature. We only recall some results with respect to the regularity of the solution **u**, which are necessary in showing that (5.8)-(5.11) satisfy Assumption 3.1-3.5 and Assumption 4.5 such that there exists a unique solution for every equation system, and the numerical scheme shown in the last chapter is useful in solving the problems.

Let $H^s := W^{s,2}$, $H := L^2$ and H^1_c denotes the subspace of H^1 made of functions whose trace vanishes on the boundary. The regularity of **u** in [23, 24] is shown as follows

Theorem 5.1. *Let* $\mathbf{u}_0 \in H^2(\Omega_f)$ *and* $\mathbf{f}$ *be smooth enough (e.g.,* $\mathbf{f} \in L^\infty(\Omega_f)$ *and* $\nabla \mathbf{f} \in L^\infty(\Omega_f)$*). Assume that*

i) for the test functions of the Navier-Stokes problem (5.7) $\mathbf{v} \in H^1_c(\Omega_f), q \in L^2(\Omega_f)$ *and a given* $\mathbf{g} \in L^2(\Omega_f)$*, the steady Stokes problem*

$$-\Delta \mathbf{v} + \nabla q = \mathbf{g}, \quad \nabla \cdot \mathbf{v} \text{ in } \Omega_f, \quad \mathbf{v}|_{\partial\Omega} = \mathbf{0},$$

has a unique solution that satisfies the inequality

$$\|\mathbf{v}\|_2 + \|q\|_1 \leq c\|\mathbf{g}\|,$$

where c is a constant;

ii) there exists a time T, $0 < T < \infty$ *and a constant* C_1 *such that the weak solution* *of the Navier-Stokes problem* (5.7) *satisfies*

$$\sup_{0<t<T} \|\nabla \mathbf{u}(t,\cdot)\| \leq C_1.$$

Then there exists a constant C_2 such that the weak solution of the Navier-Stokes problem (5.7) *satisfies*

$$\sup_{0<t<T} \|\mathbf{u}(t,\cdot)\|_2 \leq C_2.$$

Given the hypotheses of Theorem 5.1 hold, if $\mathbf{g}$ and $\mathbf{f}$ are more regular, we have the following results with respect to the regularity of $\mathbf{u}$

Theorem 5.2. *Suppose that the hypothesis of Theorem 5.1 hold. Let Ω be any two-dimensional domain whose boundary is uniformly of class C^2. Suppose that the boundary value prescribed can be extended to a solenoidal function $\mathbf{g} \in C^\infty(\overline{\Omega})$ and that $\mathbf{f} \in C^\infty((0,\infty) \times \overline{\Omega})$. Then on some interval $[0,T]$, there exists a solution $\mathbf{u}, P$ of the Navier-Stokes problem such that, in particular, $\mathbf{u} \in C((0,T) \times \overline{\Omega}) \cap C^\infty((0,T) \times \Omega)$.*

Since $\boldsymbol{\mu}_f$ has the from (5.3), with $\mathbf{u}$ being regular as stated in Theorem 5.2, the vector function $\boldsymbol{\mu}_f$ is bounded, i.e. there exists a constant $C > 0$, such that

$$|\boldsymbol{\mu}(t,\cdot)| \leq C, \quad \text{for all } t > 0, \text{ in } \Omega,$$

and

$$\mu_{ij} \in C((0,T) \times \overline{\Omega}) \cap C^\infty((0,T) \times \Omega).$$

Moreover, $\boldsymbol{\mu}$ is symmetric and positive definite. Hence the assumptions in Assumption 3.1 and 3.2 are satisfied.

If we assume that $\boldsymbol{\mu}$ does not depend on $\mathbf{u}$, i.e., the coefficient $\beta = 0$ in (5.3), it is enough for $\mathbf{u} \in H^{\frac{3}{2}}(\Omega_f)$ such that $\boldsymbol{\sigma}(\mathbf{u}) \in L^2(\Gamma)$. Together, with a suitable linear mapping $\mathcal{M}$, the boundary condition given in (5.11) satisfies Assumption 3.4 and 4.5.

The results with respect to the solutions to (5.8)-(5.10) are as follows.

Lemma 5.3. *If the solution $\mathbf{u}$ of the Navier-Stokes problem* (5.7) *is smooth enough, i.e., as given in Theorem 5.2, the space-time noise $\xi \in L^2(\Gamma)$, and the linear mapping $\mathcal{M} : [0,T] \times H \to L^2(\Gamma)$ is strongly measurable and adapted, as well as Lipschitz and of linear growth in H uniformly in $[0,T]$, such that the boundary noise $\dot{W} \in$*

$H^{-\epsilon}(\Gamma)$, $\epsilon \in (0, \frac{1}{2})$, then the Neumann boundary condition given in (5.11) *satisfies Assumption 3.4 and 4.5 such that there exists a unique solution $C_f, C_w \in \dot{H}^r$, $r \in (0, \frac{1}{2} - \epsilon)$ for the problems* (5.8)-(5.10) *respectively. Moreover, the convergence rates of Galerkin FEM for the problems are optimal.*

Proof. We assume that the covariance operator $Q \in L(L^2(\Gamma), L^2(\Gamma))$ of the space-time noise ξ is compact. Then there exists a series of positive numbers $(\lambda_j)_{j\geq 1}$ and an orthonormal basis $(e_j)_{j\geq 1}$ in $L^2(\Gamma)$ such that $\{\lambda_j, e_j\}_{j\geq 1}$ is an eigensystem of Q, i.e.,

$$Q = \sum_{j=1}^{\infty} \lambda_j e_j \otimes e_j.$$

Furthermore, the trace of the covariance operator $\mathrm{Tr}(Q) = \sum_{j=1}^{\infty} \lambda_j < \infty$ such that ξ is $L^2(\Gamma)$-valued.

As shown in the beginning of Chapter 3, the noise ξ has the form

$$\xi = \mathcal{I} W_{H_2},$$

where W_{H_2} is a H_2-cylindrical Wiener process with $H_2 = L^2(\Gamma)$ and the operator $\mathcal{I}$ belongs to $L(L^2(\Gamma), L^2(\Gamma))$. We observe that the operator $\mathcal{I} = \sum_{j=1}^{\infty} \lambda_j^{\frac{1}{2}} e_j \otimes e_j$ and Corollary 2.15 implies that $\mathcal{I} \in \gamma(L^2(\Gamma), L^2(\Gamma))$.

Thus the boundary noise $\dot{W}$ in (5.11) is written as

$$\dot{W} = \mathcal{M}(q_w - C_w)\mathcal{I} W_{H_2}.$$

We denote $\bar{\mathcal{M}} := \mathcal{M}\mathcal{I}$. With the embedding $L^2(\Gamma) \hookrightarrow H^{-\epsilon}(\Gamma)$ and Proposition 2.14, we have

$$\left\| \bar{\mathcal{M}}(t, h_1) - \bar{\mathcal{M}}(t, h_2) \right\|_{\gamma(H_2, H^{-\epsilon}(\Gamma))} \leq C_1 \left\| \bar{\mathcal{M}}(t, h_1) - \bar{\mathcal{M}}(t, h_2) \right\|_{\gamma(H_2, L^2(\Gamma))},$$

for $h_1, h_2 \in H$ and $0 < C_1 < \infty$ being constant. Moreover, for $v \in H_2$ and $x \in \Omega_f$ we have

$$\begin{aligned} \left| ((\bar{\mathcal{M}}(t, h_1) - \bar{\mathcal{M}}(t, h_2))v)(x) \right| &= \left| ((\mathcal{M}(t, h_1) - \mathcal{M}(t, h_2))(x) \right| \left| \mathcal{I} v(x) \right| \\ &\leq C_2 \left| \mathcal{M}(t, h_1)(x) - \mathcal{M}(t, h_2)(x) \right| \|v\|_{H_2}, \end{aligned}$$

where the constant $C = \|\mathcal{I}\|_{L(H^2, L^2(\Gamma))}$. Then Theorem 2.13 and the linear growth assumption of $\mathcal{M}$ implies that

$$\left\|\bar{\mathcal{M}}(t,h_1) - \bar{\mathcal{M}}(t,h_2)\right\|_{\gamma(H_2,L^2(\Gamma))} \le C_2 \left\|\mathcal{M}(t,h_1) - \mathcal{M}(t,h_2)\right\|_{L^2(\Gamma)}$$
$$\le C_2 C_3 \|h_1 - h_2\|.$$

With Proposition 2.14 and (3.3), we verify that the boundary noise $\dot{W}$ satisfies Assumption 3.4 and 4.5.

With $q_w = C_w$ for the equations (5.9) and (5.10), we define the product space $\dot{\mathbf{H}}^r := \dot{H}^r(\Omega_f) \times \dot{H}^r(\Omega_w)$, which is endowed with the norm

$$\|\mathbf{h}\|_r = \left(\|h_w\|_r^2 + \|h_w\|_r^2\right)^{\frac{1}{2}}.$$

Then the problems (5.9) and (5.10) can be written as an equation system with respect to the unknown vector $\mathbf{C} = [C_f, C_w]$,

$$\frac{\partial \mathbf{C}}{\partial t} - \mathbf{A} \cdot \mathbf{C} = \boldsymbol{\delta} \cdot \mathbf{F}, \tag{5.12}$$

where $\mathbf{A} = [\nabla \cdot (\boldsymbol{\mu}_f \nabla), \nabla \cdot (\mu_w \nabla)]$, $\mathbf{F} = [f_f, f_w] - [\mathbf{u} \cdot \nabla, 0] \cdot \mathbf{C}$ and $\boldsymbol{\delta} = [1, 1]$. The equivalent boundary condition for (5.12) is

$$\mathbf{B} \cdot \mathbf{C} = 2(\tilde{\zeta} + \dot{W}),$$

where $\mathbf{B} = [\mathbf{n} \cdot \boldsymbol{\mu}_f \nabla, \mathbf{n} \cdot \mu_w \nabla]$. Hence the proof for the problem (5.12) can be shown in a similar way.

Thus, together with Assumption 3.1 and 3.2 satisfied by the smoothness of $\mathbf{u}$, there exists a unique solution for the problems (5.8)-(5.10) respectively.

Finally, the estimate of (4.23) shows that the Galerkin FEM converges at the optimal rates when the boundary noise $\dot{W} \in H^{-\epsilon}(\Gamma)$ with $\frac{3}{2} - \epsilon > 1 + r$, i.e., $r < \frac{1}{2} - \epsilon$.

$\square$

5.4 Numerical results

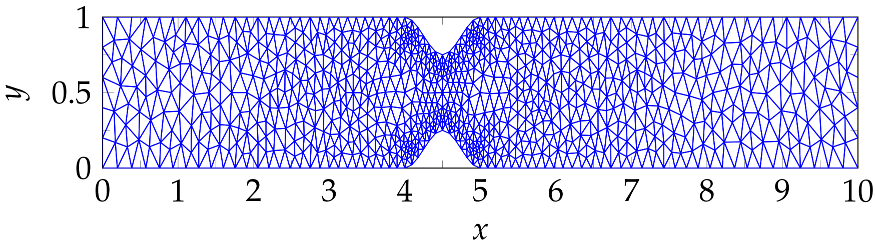
The computational domains of two models in Figure 5.1 are generated with the functions below:

$$\Gamma = \begin{cases} 0 \leq x < 4 : y = Y, \\ 4 \leq x \leq 5 : y = \left| Y - \frac{R}{4}\left(1 + \cos\left(2\pi\left(x - \frac{1}{2}\right)\right)\right)\right|, \\ 5 < x \leq 10 : y = Y, \end{cases}$$

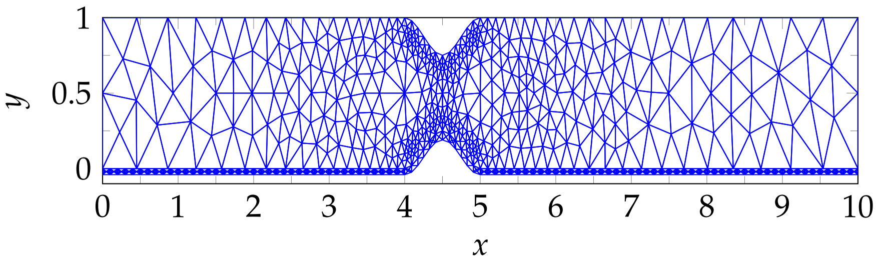
$$\Gamma_{\text{out}} = \begin{cases} 0 \leq x < 4 : y = -0.02, \\ 4 \leq x \leq 5 : -0.02 + \frac{R}{4}\left(1 + \cos\left(2\pi\left(x - \frac{1}{2}\right)\right)\right), \\ 5 < x \leq 10 : y = -0.02, \end{cases}$$

where $Y = 0$ for the bottom boundary and $Y = 1$ for the top boundary, and $R = 0.5$ is the radius of lumen.

Then the meshes of domains are generated by Gmsh [21] accordingly as shown in Figure 5.2. Compare to the mesh of the *wall-free* model (Figure 5.2a), the mesh of the *fluid-wall* model (Figure 5.2b) has a thin layer of wall on the bottom. Since the computational domain of the blood vessel defined in Figure 5.2a is symmetric with respect to the horizontal axis, to consider the vessel wall only on one side and keep the other side as wall-free in Figure 5.2b simplifies the problem without losing the generality.



(a) Mesh of the wall-free model

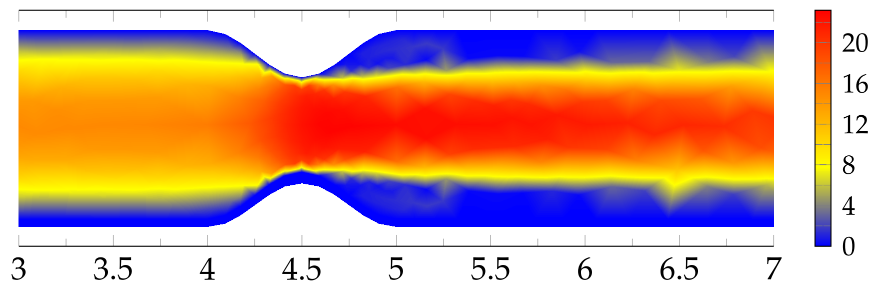


(b) Mesh of the fluid-wall model

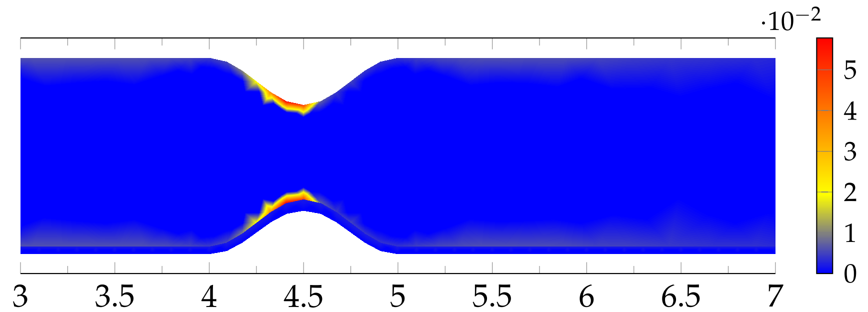
Figure 5.2: Meshes of two models

In the implementation of the numerical scheme, we adopt the varieties of Galerkin methods in splitting the problems and stabilizing the numerical scheme. To split the velocity and pressure problem in the Navier-Stokes equation (5.7), we carry out so-called *Yosida method* as shown in [36]. To achieve the computation efficiency, the problem (5.12) is split into two subproblems in the two subdomains, i.e., the lumen Ω_f and the wall Ω_w. Then the subproblems are solved in an iterative framework, namely *iterative substructuring methods*. The details of the specific technique can be found in [35]. The relevant issues with respect to the algorithms were also considered at great length in [37]. As said at the beginning of the thesis, the individual and specific numerical technique is not our focus. We intend to show that the SPDEs with the boundary noise are useful in modeling and solving real world problems. Especially they shine a light on the quantitative interpretation of the results.

The velocity of the blood $\mathbf{u}$ is same for both models since the blood speed is trivial in the vessel walls. The Navier-Stokes equation (5.7) admits a stationary solution. The stationary state is tested by the L^2-norm $\frac{\|\mathbf{u}_n - \mathbf{u}_{n-1}\|}{\|\mathbf{u}_n\|} < 10^{-4}$ at every time step. The solutions of the Navier-Stokes equation (5.7), i.e., the velocity of the blood $\mathbf{u}$ in the stationary state and the blood pressure P is shown in the figure below:



(a) Velocity in the stationary state



(b) Blood pressure

Figure 5.3: Velocity and pressure of the blood flow

Besides the solution $\mathbf{u}$, we compute the shear pressure $\boldsymbol{\sigma}(\mathbf{u})$ on the boundary Γ at each time step.

We know that the solute concentration in the vessel C_f and that outside of the vessel q_w achieves equilibrium asymptotically. Hence the total solute flux leaving Ω_f over Γ defined by the equation:

$$F(t) = \int\limits_{\Gamma} \zeta(C_f - q_w),$$

where $q_w = \kappa_w$ for the *wall-free* model and $q_w = C_w$ for the *fluid-wall* model, shows not only the solute dynamic w.r.t. the time but also the state of equilibrium when it is achieved. The total solute flux for the wall-free model is presented in Figure 5.4. It can be seen that the boundary noise does not affect when the total solute flux achieves equilibrium. The total solute flux reaches the stable state almost the same time for the deterministic boundary case and boundary noise case.

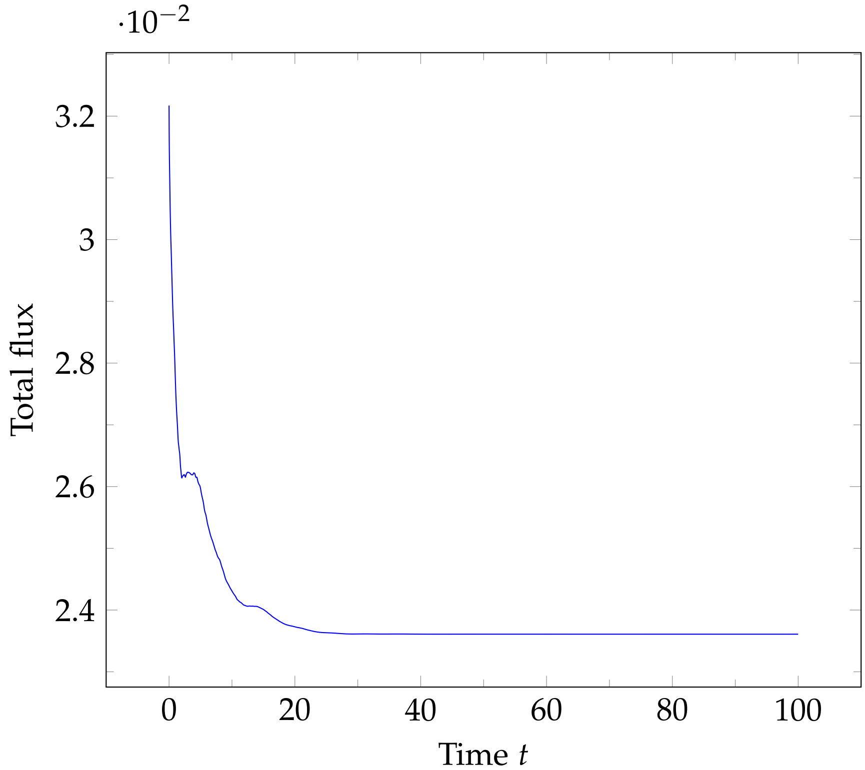
However, modeling the solute dynamic with SPDEs changes how the state of equilibrium is defined. For the model with the deterministic boundary, the condition of achieving equilibrium is that there is a time $t_E > 0$ such that for any $t_n > t_E$, the flux $F(t_n) = F(t_E)$. Thus the equilibrium state of the total flux is a "number". Quite differently, the equilibrium state for the model with the boundary noise is defined with respect to a "distribution". That is, $F(t_n)$ is a random variable denoted by X such that its density function $f_X(x)$ remains the same for any $t_n > t_E$.

We observe the same for the fluid-wall model, the total flux of which is shown in Figure 5.5.

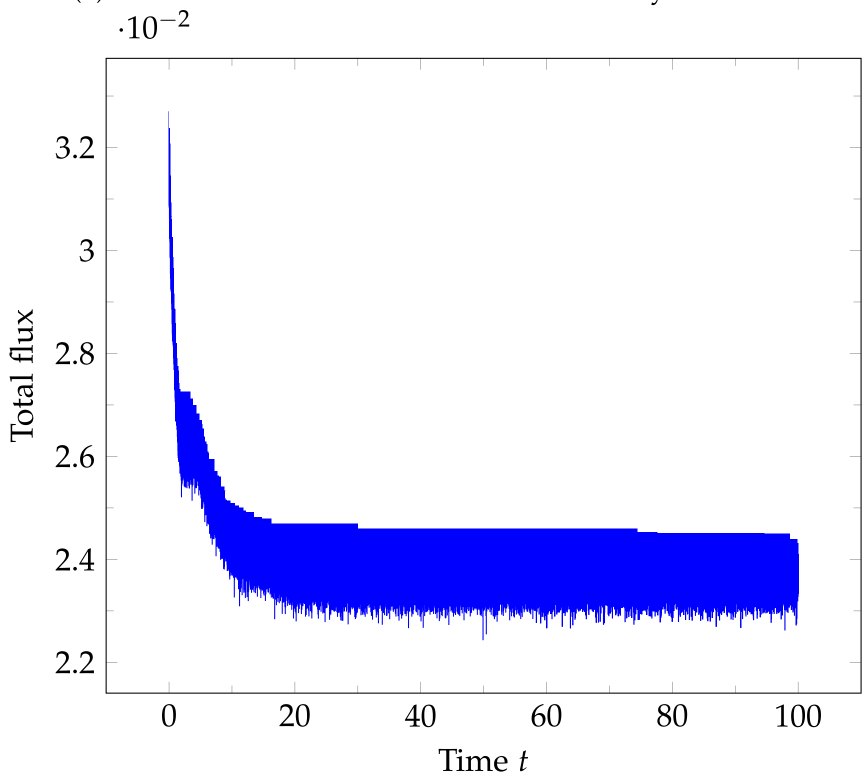
As it is shown in Figure 5.4b and Figure 5.5b, the oxygenation of vessel walls in the fluid-wall model is still lower than that in the wall-free model. It is consistent with what happens for the deterministic boundary cases and the results in [37].

In conclusion, we have made an extension of two models considered in [37] to SPDEs, with noise entering from the boundary. It is based on the pathology of how concentrates in the blood are absorbed by the organs and tissues through the arteries walls. The permeability of the membranes influences when and how the total flux achieves equilibrium state. In the case of the solute being oxygen, the models may be of great help in the investigation of the pathology of artery trees.

By introducing SPDEs with boundary noise, the permeability is given a more practical and experimental sense. In that, the permeability is essentially an experimental estimate and we acknowledge the factors that we know for sure that it depends on, e.g., the shear pressure. It is said in [37] that one particular difficulty in making quantitative sense of the total flux or the equilibrium state is that it depends significantly on the permeability. Hence more refined experimental estimates of the permeability is required to allow precise physiological evaluations.

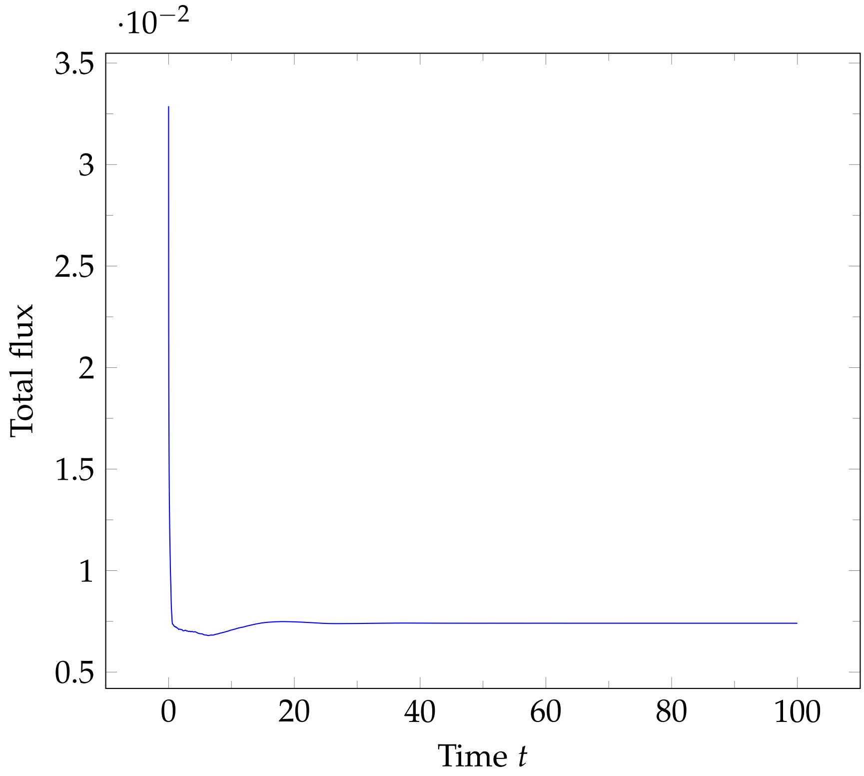


(a) Total flux with the deterministic boundary condition

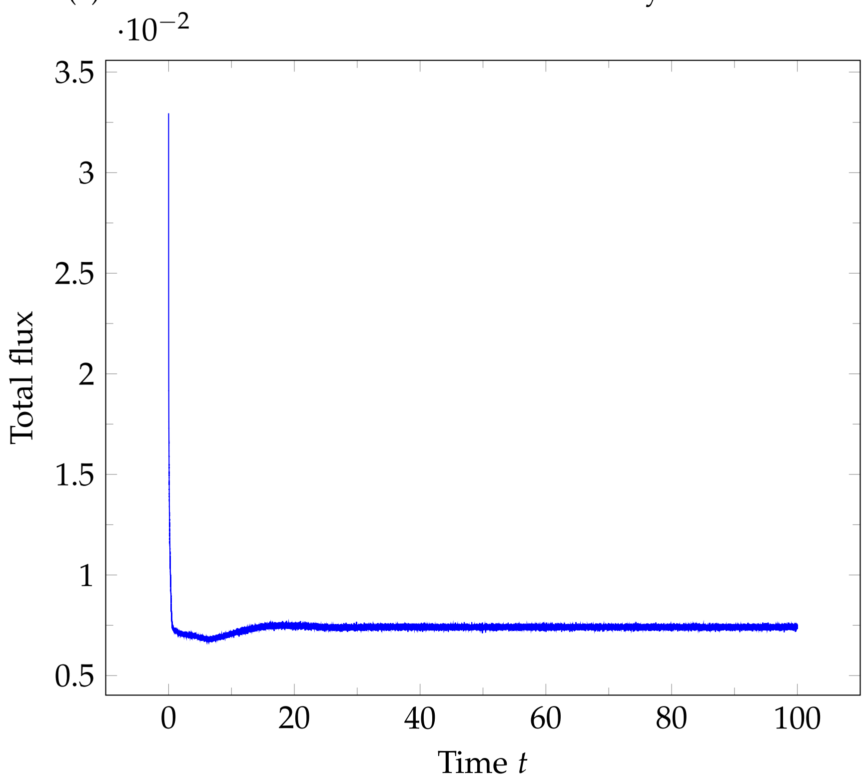


(b) Total flux with the boundary noise

Figure 5.4: Total flux for the wall-free model



(a) Total flux with the deterministic boundary condition



(b) Total flux with the boundary noise

Figure 5.5: Total flux for the fluid-wall model

Chapter 6

Heat equation with Dirichlet white-noise boundary conditions

In this chapter, we are concerned with the heat equation with the Dirichlet white-noise conditions given by

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= \Delta U(t,x) \qquad \text{on } [0,T]\times\mathbb{R}_+, \\
U(t,0) &= \dot{W}_t \qquad \text{on } [0,T], \\
U(0,x) &= u_0 \qquad \text{on } \mathbb{R}_+.
\end{aligned}
\tag{6.1}
$$

The problem is notorious due to its instantaneous blow-up near the boundary $\partial D = \{0\}$ such that the solution does not exist in a Sobolev space but a weighted Sobolev space. The existence and the properties of solution to (6.1) were well studied in the literature. For example, the continuity and boundary behaviors were studied in [1] and [39]. A thorough review of this type of SPDEs was given in [40]. So far there lacks literature in the numerical solutions of the equations with white-noise Dirichlet boundary conditions. We are interested and intend to answer the following questions in regard to the numerical solutions of (6.1):

1. How irregular can the solution be while a convergence of the numerical scheme is still satisfied?

2. What is the convergence rate of the numerical scheme?

3. What are the constraints of the convergence rate?

6.1 Solutions in weighted Sobolev space

It has been shown in [34] that the solution to (6.1) cannot exist in the space $L^2(D)$. However, it uniquely exists in an appropriate weighted Sobolev space denoted by L^p_ϱ, which is the space such that for any real-valued function $f \in L^p_\varrho$, satisfies

$$\int_{\mathbb{R}_+} |f(x)|^p (x^{p-1+\varrho} \wedge 1) dx < \infty$$

where $0 < \varrho < 1$ and $p \geq 2$. We recall several important results regarding to the solution to (6.1).

In [1], more general nonlinear SPDEs are considered given as

$$\begin{aligned} \frac{\partial U}{\partial t}(t,x) &= \Delta U(t,x) + \sum_{j=1}^{n} \left[b_j(x) \frac{\partial U}{\partial x}(t,x) + F_j(t,x,U(t,x)) \right] \dot{V}_t^j \\ U(t,0) &= \dot{W}_t \\ U(0,x) &= 0. \end{aligned} \tag{6.2}$$

The paper shows that the solution to (6.2) is given by

$$\begin{aligned} U(t,x) = &\int_0^t \frac{\partial p_D}{\partial y}(s,t,0,x) dW_s \\ &+ \sum_{j=1}^{n} \int_0^t \left(\int_{\mathbb{R}_+} p_D(s,t,y,x) F_j(t,x,U(t,x)) dy \right) dV_s^j, \end{aligned}$$

where $p_D(s,t,y,x)$ is the fundamental solution of the linear homogeneous part of (6.2). Given the additive noise V_t^j in (6.2), the fundamental solution $p_D(s,t,y,x)$ is an adapted process, which satisfies

$$\begin{aligned} p_D(s,t,y,x) =& q_D(s,t,y,x) \\ &+ \sum_{j=1}^{n} \int_s^t \left(\int_{\mathbb{R}_+} b_j(z) q_D(r,t,z,x) \frac{\partial p_D}{\partial z}(s,r,y,z) dz \right) dV_r^j, \end{aligned}$$

where $q_D(s,t,y,x)$ is the heat kernel on $\mathbb{R}_+$ defined by the Laplace operator Δ with the homogeneous Dirichlet boundary conditions.

Thus, the solution to (6.1) satisfies

$$U(t,x) = \int_0^t \frac{\partial p_D}{\partial y}(s,t,0,x)dWs + \int_{\mathbb{R}_+} p_D(0,t,y,x)u_0 dy. \tag{6.3}$$

The fundamental solution $p_D(s,t,y,x)$ can be simplified to the heat kernel $q_D(s,t,y,x)$ and coincides with the semi-group operator (see [39, Chapter 5]). The stochastic integral in (6.3) is interpreted in the backward Itô sense as shown in Chapter 2.

We have two important estimates of $p_D(s,t,y,x)$ as follows:

Lemma 6.1. *[1, Lemma 6] For all $s < t$, $x, y \in \mathbb{R}_+$, it holds that*

$$|p_D(s,t,y,x)| \leq C(t-s)^{-\frac{1}{2}} \exp\left(-\frac{|y-x|^2}{c(t-s)}\right),$$

$$\left|\frac{\partial^{m+k} p_D}{\partial y^k \partial x^m}(s,t,y,x)\right| \leq C(t-s)^{-\frac{m+k+1}{2}} \exp\left(-\frac{|y-x|^2}{c(t-s)}\right),$$

for each $m = 0, 1, 2$, $k = 0, 1$ and for some constant $C, c > 0$.

The next lemma is concerned with the upper bounds of the time derivative of $p_D(s,t,y,x)$. Since $p_D(s,t,y,x)$ is the stochastic kernel in [1], the *forward Malliavin derivative* is introduced. It is not necessary for the heat kernel in our case. Hence the classic estimate of time derivative of the heat kernel is enough.

Lemma 6.2. *[15, Theorem 3]([1, Lemma 8]) For all $0 \leq s \leq t \leq T$, $x, y \in \mathbb{R}_+$, the time derivative of $p_D(s,t,x,y)$ satisfies the estimates*

$$\left|\frac{\partial p_D}{\partial s}(s,t,y,x)\right| \leq C(t-s)^{-1} \exp\left(-\frac{|y-x|^2}{c(t-s)}\right),$$

and

$$\begin{aligned}&\left|\frac{\partial p_D}{\partial s}(s,t,y,x) - \frac{\partial p_D}{\partial s}(s,t,y,z)\right| \\ &\quad \leq (t-s)^{-1-\frac{\sigma}{2}}|z-x|^\sigma \left(\exp\left(-\frac{|y-x|^2}{c(t-s)}\right) - \exp\left(-\frac{|y-z|^2}{c(t-s)}\right)\right),\end{aligned}$$

for $0 \leq \sigma \leq 1$

The *weak solution* to (6.2) is defined in [1] as follows

Definition 6.3. A function $U = U(t,x) \in L^p(\Omega; L^p_\varrho)$, continuous in $x \in (0,+\infty)$, is a weak solution of (6.2) if the following identity holds

$$\begin{aligned}\lim_{\epsilon\to 0} \Bigg\{ \int_0^\infty \int_{\mathbb{R}_+} & U(s,x+\epsilon) \left[\frac{\partial \varphi}{\partial s}(s,x) + \frac{\partial^2 \varphi}{\partial x}(s,x) \right] dxds \\ &+ \sum_{j=1}^n \int_0^\infty \int_{\mathbb{R}_+} F_j\left(s, x+\epsilon, U(s,x+\epsilon)\right) \varphi(s,x) dx dV_s^j \\ &- \sum_{j=1}^n \int_0^\infty \int_{\mathbb{R}_+} \frac{\partial}{\partial x}[b_j(x+\epsilon) f(s,x)] u(s,x+\epsilon) dx dV_s^j \Bigg\} = - \int_0^\infty \frac{\partial \varphi}{\partial x}(s,0) dW_s,\end{aligned}$$

where $\varphi(t,x) \in C_c^\infty$ is a family of smooth functions with compact support and $\varphi(t,0) = 0$.

Hence the weak solution of (6.1) has the identity

$$\lim_{\epsilon\to 0} \left\{ \int_0^\infty \int_{\mathbb{R}_+} U(s,x+\epsilon) \left[\frac{\partial \varphi}{\partial s}(s,x) + \frac{\partial^2 \varphi}{\partial x}(s,x) \right] \right\} = - \int_0^\infty \frac{\partial \varphi}{\partial x}(s,0) dW_s. \quad (6.4)$$

As stated in [1, Theorem 3], the solution given by the evolution equation (6.3) is also the unique weak solution of (6.1) given by (6.4).

To apply Galerkin FEM, we consider the solution existing in a weighted Sobolev space $W_\varrho^{2,r}$, denoted by H_ϱ^r with $0 \leq r < 1$, consequently $H_\varrho := L_\varrho^2$. We recall the result in Chapter 2 that H_ϱ^r is a Hilbert space. Since the convergence rate depends the regularity or the order of the Hilbert space, the next result of gives the regularity of the solution.

Theorem 6.4. *For $r \in [0,1)$ and $p \geq 2$, if $\varrho > 4r$ and $U(0) \in H_\varrho^r$, then the unique solution $U \in L^p(\Omega; H_\varrho)$ to (6.1) satisfies*

$$P\left(U(t) \in H_\varrho^r\right) = 1, \quad t \in [0,T]$$

Proof. By taking norms of (6.3) and applying Proposition 2.3, we have

$$\|U(t)\|_{L^p(\Omega;H^r_\varrho)} \leq \|U(t)\|_{L^p(\Omega;H_\varrho)} + C\left\|\int_0^t (-\Delta)^{\frac{r}{2}}\frac{\partial p_D}{\partial y}(s,t,0,x)dW_s\right\|_{L^p(\Omega;H_\varrho)} + C\left\|\int_0^1 (-\Delta)^{\frac{r}{2}} p_D(0,t,y,x)u_0 dy\right\|_{L^p(\Omega;H_\varrho)}$$

The first term $\|U(t)\|_{L^p(\Omega;H_\varrho)} < \infty$. Let $\partial H := H(\partial D)$ with $\partial D = \{0\}$ be a separable Hilbert space, i.e., $\partial H = \mathbb{R}$, the second term

$$\begin{aligned} II &:= C\left\|\int_0^t (-\Delta)^{\frac{r}{2}}\frac{\partial p_D}{\partial y}(s,t,0,x)dW_s\right\|_{L^p(\Omega;H_\varrho)} \\ &\leq C\left\{E\left|\int_0^t \left\|(-\Delta)^{\frac{r}{2}}\frac{\partial p_D}{\partial y}(s,t,0,x)\right\|^2_{\gamma(\partial H,H_\varrho)} ds\right|^{\frac{p}{2}}\right\}^{\frac{1}{p}}, \end{aligned}$$

since $(-\Delta)^{\frac{r}{2}}\frac{\partial p_D}{\partial y}(s,t,0,x)$ is the γ-*radonifying* mapping ∂H to H_ϱ ([39, Theorem 5.10]). Let φ_m denote an orthogonal basis of ∂H, with Lemma 6.1, we have

$$\begin{aligned} &\int_0^t \sum_{m=1}^\infty \left\|(-\Delta)^{\frac{r}{2}}\frac{\partial p_D}{\partial y}(s,t,0,x)\varphi_m\right\|^2_{H_\varrho} ds \\ &\leq \int_0^t \sum_{m=1}^\infty \int_0^1 \left|(-\Delta)^{\frac{r}{2}}\frac{\partial p_D}{\partial y}(s,t,0,x)\varphi_m\right|^2 x^{1+\varrho}dxds \\ &\leq C\int_0^t \sum_{m=1}^\infty \int_0^1 (t-s)^{-2-2r}\exp\left(-\frac{|x|^2}{c(t-s)}\right)|\varphi_m|^2 x^{1+\varrho}dxds \\ &\leq C\int_0^t \sum_{m=1}^\infty \int_0^1 (t-s)^{-2-2r+1+\alpha}x^{1+\varrho-2-2\alpha}|\varphi_m|^2 dxds < \infty, \end{aligned}$$

when $\varrho > 2\alpha > 4r$. Thus the second term $II < \infty$ if $\varrho > 4r$. For the last term, by Lemma 6.1 and Hölder inequality we have

$$
\begin{aligned}
III &:= C \left\| \left(\int_0^1 \left(\int_0^1 \left| (-\Delta)^{\frac{r}{2}} p_D(0,t,y,x) u_0 \right| dy \right)^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\
&\leq C t^{-\frac{1}{2}-\frac{1+\varrho}{4}} \left\| \left(\int_0^1 \left(\int_0^1 \left| \exp\left(-\frac{|y-x|^2}{ct} \right) y^{\frac{1+\varrho}{2}} (-\Delta)^{\frac{r}{2}} u_0 \right| dy \right)^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\
&\leq C t^{-\frac{1}{2}-\frac{1+\varrho}{4}} \left\| \left(\int_0^1 \left(\int_0^1 \exp\left(-\frac{|y-x|^2}{ct} \right) dy \right) \|u_0\|^2_{H^r_\varrho} x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\
&\leq C t^{-\frac{1}{2}-\frac{1+\varrho}{4}} \left(\int_0^1 \left(\int_0^1 \exp\left(-\frac{|y-x|^2}{ct} \right) dy \right) x^{1+\varrho} dx \right)^{\frac{1}{2}}.
\end{aligned}
$$

We recall the estimate of the integral

$$
\int_0^1 y^b \exp\left(-\frac{|y-x|^2}{ct} \right) dy \leq C|x|^{-2} t, \tag{6.5}
$$

when $b > -1$. Hence $III < \infty$. Altogether the proof is completed.

□

From the result above, we note that the solution can exist in a more regular space $H^r_\varrho \subset H_\varrho$. However, the regularity is constrained by the weight power ϱ, this means that the convergence rate of the numerical scheme will also be constrained by ϱ. In the following sections of the chapter, we will see that not only the existence of solution require an upper limit $\varrho < 1$ but also the numerical scheme has such requirement to converge.

6.2 Discontinuous Galerkin time stepping

In previous chapters we have considered a numerical scheme with Galerkin FEM in space and the backward Euler stepping in time for the semi-linear SPDEs with the Neumann boundary noise. We have proved that under certain assumptions, the numerical scheme has the optimal convergence rate as shown in **Theorem 4.9**. However, for the equations with the Dirichlet white boundary noise, we are facing several challenges that cause the same numerical scheme to no longer work:

1. The boundary operator cannot be used to map the boundary condition into domain since the solution or the trace of the solution does not exist on the boundary;

2. As a consequence, neither Assumption 3.4 nor Assumption 4.5 is valid, i.e, no extra temporal regularity can be assumed;

3. There are difficulties to look for the rational approximation of semigroups since we do not have an equation with homogeneous boundary condition equivalent to (6.1).

As shown in the proof of **Theorem 4.9**, not only Assumption 3.4 but also Assumption 3.2 and Assumption 3.3 are essential. Without the extra temporal regularity, the numerical scheme could explode due to the non-smooth spatial data. Especially we cannot decide the order of the accuracy of backward finite difference in time in this case. Hence we look into a temporal discretization that requires minimum temporal regularity, i.e., the discontinuous Galerkin time stepping scheme.

Similar to Galerkin FEM in space, the discontinuous Galerkin time stepping relaxes the regularity requirement of the solution in time. With this method, the solution is only required to be smooth locally but not globally. Thus we can look at a "richer" space to find the solution. Let H be an abstract separable Hilbert space and assume that A is a self-adjoint, positive definite operator, not necessarily bounded but with compact inverse, and $\mathcal{D}(A) \subset H$. We consider the initial value problem

$$\begin{aligned} u' + Au &= f, \quad \text{for } t \in (0, T], \\ u(0) &= u_0. \end{aligned} \tag{6.6}$$

The exact solution of (6.6) exists in a weak sense not only on space but in time. That is, for a smooth function with the compact support $v \in C_c^\infty(D)$,

$$\int_0^T \left((u', v) + (u, Av)\right) dt = \int_0^T (f, v) dt.$$

After integration by parts of the first term, we have

$$\int_0^T \big(-(u, v') + (u, Av)\big)\, dt = (u_0, v(0)) + \int_0^T (f, v) dt, \tag{6.7}$$

when $v(T) = 0$. Note that it has the same form as the weak solution of stochastic heat equation given by (6.4).

Now we discretize the problem (6.7) temporally by partitioning $[0, T]$ by $0 = t_0 < t_1 \cdots < t_j \cdots < t_N = T$ and denote $\Xi_j = (t_{j-1}, t_j]$ and $k = \max(t_j - t_{j-1})$. Hence the problem is reduced to looking for an approximate solution to (6.7) in each Ξ_j. The solution is a polynomial of t of degree at most $q-1$ with the coefficient in H and belongs to the space

$$\mathcal{S}_k = \left\{ V : [0, \infty) \to H; V \mid_{\Xi_j} = \sum_{i=0}^{q-1} \varphi_i t^i, \varphi_i \in H \right\}.$$

The functions in $\mathcal{S}_k$ are not necessarily continuous at the t_j. Hence for $V \in \mathcal{S}_k$ we denote V^j and V_+^j the value of $V(t_j)$and its limit from above at t_j. For each interval Ξ_j the solution $U \in \mathcal{S}_k$ to (6.7) satisfies for all $V \in \mathcal{S}_k$

$$\left\{ \int_{\Xi_j} -(U, V') + (U, AV) \right\} - (U^j, V^j) + (U_+^{j-1}, V_+^{j-1}) \\ = (U^{j-1}, V_+^{j-1}) - \int_{\Xi_j} (f, V),$$

To apply the full discretization to (6.7), we introduce the finite dimensional space by changing the space of coefficients φ in $\mathcal{S}_k$ to the finite subspace S_h given by:

$$\mathcal{S}_{kh} = \left\{ V : [0, \infty) \to S_h; V \mid_{\Xi_j} = \sum_{i=0}^{q-1} \varphi_i t^i, \varphi_i \in S_h \right\}.$$

Now the fully discretized problem is formulated by

$$\left\{ \int_{\Xi_j} -(U_h, V') + (U_h, AV) \right\} - (U_h^j, V^j) + (U_{h+}^{j-1}, V_+^{j-1})$$

$$= (U_h^{j-1}, V_+^{j-1}) - \int_{\Xi_j} (f, V) \;\; \forall V \in \mathcal{S}_{kh}, \quad (6.8)$$

where the solution $U_h \in \mathcal{S}_{kh}$. It was shown that (6.8) has a unique solution in [45].

6.3 Error estimate of spatial semi-discretization

Similar to the Galerkin FEM scheme used for the spatial discretization of the Hilbert space H in Chapter 4, we introduce a discretized solution space with weight $S_{h,\varrho} \subset H_\varrho^1$ being a finite dimensional subspace of H_ϱ. Also, the mapping $P_{h,\varrho} : L_\varrho^2 \to S_{h,\varrho}$ is the orthogonal projection on $S_{h,\varrho}$ in L_ϱ^2. The orthogonal projector $P_{h,\varrho}$ is well-defined and the best approximation in the L_ϱ^2-norm (see [8] and [32]):

$$\|x - P_{h,\varrho}x\|_{0,\varrho} \le Ch^\mu \|x\|_{\mu,\varrho}, \quad \mu \ge 0, \; \forall x \in H_\varrho^\mu. \tag{6.9}$$

Hence, the spatially discretized formulation of (6.1) is given by

$$\begin{aligned} \frac{\partial U_h}{\partial t} &= \Delta U_h \quad \text{on } [0,T] \times \mathbb{R}_+ \\ U(t,0) &= \dot{W}_t \quad \text{on } [0,T] \\ U(0,x) &= P_{h,\varrho} u_0 \quad \text{on } \mathbb{R}_+, \end{aligned} \tag{6.10}$$

To simplify the error estimate without raising major issues, we truncate the spatial domain $\mathbb{R}_+$ to $[0,1]$ with $U(t,1) = 0$. Then we have the following error estimate for the spatial semi-discretization:

Theorem 6.5. *For $r \in [0,1)$ and $p \in [2,\infty)$, with $U(t) \in L^p(\Omega; H_\varrho^r)$, there exists a constant C, independent of $h \in (0,1]$, such that*

$$\|U_h(t) - U(t)\|_{L^p(\Omega;H_\varrho)} \le Ch^r, \;\; \forall t \in (0,T],$$

where $U_h(t)$ and $U(t)$ are the solutions of (6.10) and (6.1)respectively.

Proof. The error is written as

$$U_h(t) - U(t) = U_h(t) - P_{h,\varrho}U(t) + P_{h,\varrho}U(t) - U(t) =: \theta(t) + \rho(t).$$

Hence the Cauchy inequality gives

$$\|U_h(t) - U(t)\|_{L^p(\Omega;H_\varrho)} \leq \|\theta(t)\|_{L^p(\Omega;H_\varrho)} + \|\rho(t)\|_{L^p(\Omega;H_\varrho)}.$$

Since $U_h(t)$ satisfy (6.4), we have

$$\lim_{\epsilon\to 0}\left\{\int_0^\infty\int_{\mathbb{R}_+} U_h(s,x+\epsilon)\left[\frac{\partial\varphi}{\partial s}(s,x) + \frac{\partial^2\varphi}{\partial x}(s,x)\right]\right\} = -\int_0^\infty \frac{\partial\varphi}{\partial x}(s,0)dW_s. \quad (6.11)$$

The subtraction between (6.11) and (6.4) gives:

$$\lim_{\epsilon\to 0}\left\{\int_0^\infty\int_{\mathbb{R}_+} \left[U_h(s,x+\epsilon) - P_{h,\varrho}U(s,x+\epsilon) + P_{h,\varrho}U(s,x+\epsilon) - U(s,x+\epsilon)\right]\right.$$
$$\left.\left[\frac{\partial\varphi}{\partial s}(s,x) + \frac{\partial^2\varphi}{\partial x}(s,x)\right]\right\} = 0.$$

With the orthogonal projector $P_{h,\varrho}$ satisfying

$$(P_{h,\varrho}x, y_h)_\varrho = (\Delta^{-1}x, y_h)_{1,\varrho}, \quad \forall x \in H_\varrho^r, y_h \in S_{h,\varrho},$$

we have

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^+} \left[P_{h,\varrho}U(s,x+\epsilon) - U(s,x+\epsilon)\right]\frac{\partial^2\varphi}{\partial x}(s,x)(x+\epsilon)^{1+\varrho} = 0.$$

Thus,

$$\lim_{\epsilon\to 0}\left\{\int_0^\infty\int_{\mathbb{R}_+} \theta(t)\left[\frac{\partial\varphi}{\partial s}(s,x) + \frac{\partial^2\varphi}{\partial x}(s,x)\right] + \rho(t)\frac{\partial\varphi}{\partial s}(s,x)\right\} = 0.$$

It is the weak solution of the PDE with respect to θ:

$$\begin{aligned}
\frac{\partial\theta}{\partial t} &= \Delta\theta - \frac{\partial\rho}{\partial t}, \quad \text{on } [0,T]\times(0,1),\\
\theta(t,0) &= \theta(t,1) = 0, \quad \text{on } [0,T],\\
\theta(0,x) &= (P_{h,\varrho} - \mathrm{Id})u_0, \quad \text{on } (0,1),
\end{aligned}$$

and the correspondent evolution equation is given by

$$\theta(t,x) = \int_0^1 p_D(0,t,y,x)(P_{h,\varrho} - \mathrm{Id})u_0 dy - \int_0^t \int_0^1 p_D(s,t,y,x)\frac{\partial \rho}{\partial s}(s,y)dyds.$$

Hence, with the Cauchy inequality, we have

$$\begin{aligned} \|\theta\|_{L^p(\Omega;H_\varrho)} &\le \left\| \int_0^1 p_D(0,t,y,x)(P_{h,\varrho} - \mathrm{Id})u_0 dy \right\|_{L^p(\Omega;H_\varrho)} \\ &\quad + \left\| \int_0^t \int_0^1 p_D(s,t,y,x)\frac{\partial \rho}{\partial s}(s,y)dyds \right\|_{L^p(\Omega;H_\varrho)} \\ &=: \Phi_1 + \Phi_2. \end{aligned} \tag{6.12}$$

By the results of Lemma 6.1 and (6.9), the first term of (6.12) is estimated as

$$\begin{aligned} \Phi_1 &= \left\| \left(\int_0^1 \left(\int_0^1 \left| p_D(0,t,y,x)(P_{h,\varrho} - \mathrm{Id})u_0 \right| dy \right)^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\ &\le Ct^{-\frac{1}{2}-\frac{1+\varrho}{4}} \times \\ &\quad \left\| \left(\int_0^1 \left(\int_0^1 \left| \exp\left(-\frac{|y-x|^2}{ct} \right) y^{\frac{1+\varrho}{2}} (P_{h,\varrho} - \mathrm{Id})u_0 \right| dy \right)^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\ &\le Ct^{-\frac{1}{2}-\frac{1+\varrho}{4}} \times \\ &\quad \left\| \left(\int_0^1 \left(\int_0^1 \exp\left(-\frac{|y-x|^2}{ct} \right) dy \right) \|(P_{h,\varrho} - \mathrm{Id})u_0\|_{H_\varrho}^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\ &\le Ch^r \left(\int_0^1 \left(\int_0^1 \exp\left(-\frac{|y-x|^2}{ct} \right) dy \right) x^{1+\varrho} dx \right)^{\frac{1}{2}}. \end{aligned} \tag{6.13}$$

Thus, with the result of (6.5), we have

$$\Phi_1 \leq Ch^r. \tag{6.14}$$

Regarding the second term in (6.12), first we apply the Fubini's theorem and integration by part to have

$$\begin{aligned}
\Phi_2 \leq & \left\| \int_0^1 p_D(s,t,y,x)\rho(s,y)|_{s=0}^t \, dy \right\|_{L^p(\Omega;H_\varrho)} \\
& + \left\| \int_0^1 \int_0^t \frac{\partial}{\partial s} p_D(s,t,y,x)\rho(s,y) ds dy \right\|_{L^p(\Omega;H_\varrho)} \\
=: & \, \Phi_{2.1} + \Phi_{2.2},
\end{aligned}$$

where $f(s)|_{s=0}^t$ denotes the value of $f(t) - f(0)$.

With the result in the estimate of Φ_1 and the property of the heat kernel $\int_0^1 \lim_{t->0+} p_D(0,t,y,x)U(0,y)dy = u_0$, the estimate of $\Phi_{2.1}$ is given by

$$\begin{aligned}
\Phi_{2.1} \leq & \left\| \int_0^1 \lim_{h->0^+} p_D(t,t+h,y,x)\rho(t,y)dy \right\|_{L^p(\Omega;H_\varrho)} \\
& + \left\| \int_0^1 p_D(0,t,y,x)\rho(0,y)dy \right\|_{L^p(\Omega;H_\varrho)} \\
\leq & \, Ch^r.
\end{aligned} \tag{6.15}$$

We apply Lemma 6.2 and the similar steps in (6.13) to estimate $\Phi_{2.2}$ as follows,

$$\Phi_{2.2} \leq \int_0^t \left\| \int_0^1 \frac{\partial}{\partial s} p_D(s,t,y,x)\rho(s,y)dy \right\|_{L^p(\Omega;H_\varrho)} ds$$

$$\leq C \int_0^t (t-s)^{-1-\frac{1+\varrho}{4}} \times$$

$$\left\| \left(\int_0^1 \left(\int_0^1 \left| \exp\left(-\frac{|y-x|^2}{c(t-s)} \right) y^{\frac{1+\varrho}{2}} \rho(s,y) \right| dy \right)^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} ds$$

$$\leq C \int_0^t (t-s)^{-1-\frac{1+\varrho}{4}} \times$$

$$\left\| \left(\int_0^1 \left(\int_0^1 \exp\left(-\frac{|y-x|^2}{c(t-s)} \right) dy \right) \|\rho(s,y)\|^2_{H_\varrho} x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} ds$$

Then with the result of (6.5), we have

$$\Phi_{2.2} \leq C \sup_{s\in[0,t]} \|\rho(s)\|_{H_\varrho} \int_0^t (t-s)^{-\frac{1}{2}-\frac{1+\varrho}{4}} \left(\int_0^1 x^{-1+\varrho} dx \right)^{\frac{1}{2}} ds \leq Ch^r \tag{6.16}$$

Together (6.14) - (6.16), the proof is completed. □

6.4 Error estimate of full-discretization with the discontinuous Galerkin time stepping

Now we apply the discontinuous Galerkin time stepping on the stochastic heat equation (6.1). Similarly, we denote the solution space as

$$\mathcal{S}_{kh,\varrho} = \left\{ V : [0,\infty) \to S_{h,\varrho}; V\,|_{\Xi_j} = \sum_{i=0}^{q-1} \varphi_i t^i, \varphi_i \in S_{h,\varrho} \right\}.$$

The discrete formulation of (6.1) is given by

$$\left\{ \int_{\Xi} \int_{\mathbb{R}_+} U_h \left[\frac{\partial V}{\partial s} + \frac{\partial^2 V}{\partial x} \right] \right\} - \int_{\mathbb{R}_+} U_h^j V^j + \int_{\mathbb{R}_+} U_{h+}^{j-1} V_+^{j-1}$$
$$= \int_{\mathbb{R}_+} U_h^{j-1} V_+^{j-1} - \int_{\Xi} \frac{\partial V}{\partial x}(s,0) dW_s, \quad \forall V \in \mathcal{S}_{kh,\varrho} \tag{6.17}$$

with the solution $U_h \in \mathcal{S}_{kh,\varrho}$. Thus we have the error estimate result as follows:

Theorem 6.6. *For $r \in [0,1)$ and $p \in [2,\infty)$, with $U(t) \in L^p(\Omega; H_\varrho^r)$, there exists a constant C, independent of $h,k \in (0,1]$, such that*

$$\|U_h^{N_k} - U(t_{N_k})\|_{L^p(\Omega;H_\varrho)} \le C\left(h^r + k^{\frac{r}{2}}\right),$$

where $U(t_{N_k})$ and $U_h^{N_k}$ are the solutions to (6.1) and (6.17) respectively.

Proof. Similar to the proof of [45, Theorem 12.6], first we decompose the error as

$$U_h^j - U(t_j) = U_h^j - P_{h,\varrho}\overline{U} + P_{h,\varrho}\overline{U} - U(t_j) := \theta^j + \rho^j,$$

where $\overline{U} := \overline{U}(t)$ is the piecewise constant function in t such that $\overline{U}(t) = U(t_j)$ for $t \in \Xi_j$. Hence the error is written as

$$\|U_h^{N_k} - U(t_{N_k})\|_{L^p(\Omega;H_\varrho)} \le \|\theta^{N_k}\|_{L^p(\Omega;H_\varrho)} + \|\rho^{N_k}\|_{L^p(\Omega;H_\varrho)}.$$

We have $\|\rho^{N_k}\|_{L^P(\Omega;H_\varrho)} \le Ch^r$ as shown in Theorem 6.5. Let the norm

$$\|\rho\|_{L^P(\Omega;H_\varrho);\Xi_j} = \sup_{t\in\Xi_j} \|\rho(t)\|_{L^P(\Omega;H_\varrho)}.$$

With the result of [45, Theorem 12.6], we have

$$\|\theta^{N_k}\|_{L^P(\Omega;H_\varrho)} \le CL_{N_k} \max_{j\le N_k} \|\rho\|_{L^P(\Omega;H_\varrho);\Xi_j}, \quad L_{N_k} = (\log N_k)^{\frac{1}{2}} + 1.$$

For the estimate of $\|\rho\|_{L^p(\Omega;H_\varrho);\Xi_j}$, we have

$$\begin{aligned}\|\rho\|_{L^p(\Omega;H_\varrho);\Xi_j} &= \|P_{h,\varrho}\overline{U} - U(t)\|_{L^p(\Omega;H_\varrho);\Xi_j} \\ &\leq \|(P_{h,\varrho} - \mathrm{Id})\overline{U}\|_{L^p(\Omega;H_\varrho);\Xi_j} + \|\overline{U} - U\|_{L^p(\Omega;H_\varrho);\Xi_j}\end{aligned}$$

Again the first term is bounded by Ch^r and the second term is estimated as follows,

$$\begin{aligned}\|\overline{U} - U\|_{L^p(\Omega;H_\varrho);\Xi_j} &= \sup_{s\in\Xi_j} \|U(t_j) - U(s)\|_{L^p(\Omega;H_\varrho)} \\ &= \sup_{s\in\Xi_j} \|U(t_j - s) - U(0)\|_{L^p(\Omega;H_\varrho)} \\ &= \sup_{s\in\Xi_j} \left\| \int_0^1 \left[p_D(0, t_j - s, y, x) - \mathrm{Id}\right] u_0 dy \right\|_{L^p(\Omega;H_\varrho)} \\ &\quad + \sup_{s\in\Xi_j} \left\| \int_0^{t_j - s} \frac{\partial p_D}{\partial y}(\sigma, t_j - s, 0, x) dW_\sigma \right\|_{L^p(\Omega;H_\varrho)} \\ &=: I_1 + I_2\end{aligned}$$

since $U(t)$ is a strong Markov process by Theorem 3 [7].

For the term I_2, from the estimate of the term II in Theorem 6.4, we have

$$I_2 \leq \sup_{s\in\Xi_j} C \left(\int_0^{t_j - s} \sigma^{-1+\alpha} d\sigma \right)^{\frac{1}{2}} \leq Ck^{\frac{\alpha}{2}},$$

where $\varrho > 2\alpha > 4r$.

We have the estimate of the term I_1 as below,

$$
\begin{aligned}
I_1 &= \sup_{s\in\Xi_j}\left\| \left(\int_0^1 \left| \int_0^1 \left[p_D(0,t_j-s,y,x) - \mathrm{Id}\right] u_0 dy \right|^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\
&= \sup_{s\in\Xi_j}\left\| \left(\int_0^1 \lim_{\epsilon\to 0} \left| \int_0^1 \left[p_D(0,t_j-s,y,x) - p_D(0,\epsilon,y,x)\right] u_0 dy \right|^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\
&= \sup_{s\in\Xi_j}\left\| \left(\int_0^1 \left(\int_0^1 \int_0^{t_j-s} \left| \frac{\partial p_D}{\partial \sigma}(0,\sigma,y,x) u_0 \right| d\sigma dy \right)^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\
&= \sup_{s\in\Xi_j}\left\| \left(\int_0^1 \left(\int_0^1 \int_0^{t_j-s} \left| (-\Delta)^{-\frac{\beta}{2}} \frac{\partial p_D}{\partial \sigma}(0,\sigma,y,x) (-\Delta)^{\frac{\beta}{2}} u_0 \right| d\sigma dy \right)^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)},
\end{aligned}
$$

where $0 < \beta < 1$.

With Lemma 6.2 and Hölder inequality, we have

$$
\begin{aligned}
I_1 &\le C \sup_{s\in\Xi_j}\left\| \left(\int_0^1 \left(\int_0^1 \int_0^{t_j-s} \sigma^{-\frac{3-\beta}{2}-\frac{1+\varrho}{4}} \exp\left(-\frac{|y-x|^2}{c\sigma}\right) y^{\frac{1+\varrho}{2}} \times \right.\right.\right. \\
&\qquad \left.\left.\left. |(-\Delta)^{\frac{\beta}{2}} u_0| d\sigma dy \right)^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\
&\le C \sup_{s\in\Xi_j}\left\| \left(\int_0^1 \left(\int_0^{t_j-s} \sigma^{-\frac{3-\beta}{2}-\frac{1+\varrho}{4}} \left(\int_0^1 \exp\left(-\frac{|y-x|^2}{c\sigma}\right) dy \right)^{\frac{1}{2}} \times \right.\right.\right. \\
&\qquad \left.\left.\left. \left(\int_0^1 y^{1+\varrho} |(-\Delta)^{\frac{\beta}{2}} u_0|^2 dy \right)^{\frac{1}{2}} d\sigma \right)^2 x^{1+\varrho} dx \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}
\end{aligned}
$$

$$\leq C \sup_{s\in\Xi_j}(t_j - s)^{\frac{\beta}{2}-\frac{1+\varrho}{4}} \left\| \|u_0\|_{H^\beta_\varrho} \int_0^1 |x|^{-2} x^{1+\varrho} dx \right\|_{L^p(\Omega)}.$$

Hence when $u_0 \in H^\beta_\varrho$ with $\beta > \frac{1+\varrho}{2}$, the estimate $I_1 < Ck^{\frac{\beta}{2}-\frac{1+\varrho}{4}}$.

Since $\log(N_k) \leq CN_k^\epsilon = C\left(\frac{T}{k}\right)^\epsilon$ for an arbitrary small $\epsilon > 0$, we choose appropriate ϵ, α and β such that $\alpha - \epsilon = r$ and $\beta - \frac{1+\varrho}{2} - \epsilon = r$. Thus we have the desired result. □

6.5 Numerical experiments

We apply the numerical scheme above on the one dimensional heat equation with the Dirichlet boundary noise formally written as

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\Delta u,\ t \in (0,1], x \in (0,1) \\ u(0,x) &= 100\sin(\pi x), \\ u(t,0) &= \dot{W}(t),\ u(t,1) = 0. \end{aligned} \tag{6.18}$$

Recall that the Wiener process $W(t)$ has the form

$$W(t) = \sum_{m=1}^{\infty} \lambda_m \beta_m(t) \varphi_m, \tag{6.19}$$

where β_m are independent Brownian motions, (γ_m) is a sequence of real numbers and (φ_m) is a sequence of functions defined on the boundary ∂D when $D \subset \mathbb{R}^n, n \geq 2$. With the same argument in **Chapter 4** section 4.4, the expansion (6.19) is truncated as

$$W(t) = \sum_{m=1}^{B_h} \lambda_m \beta_m(t) \varphi_m, \tag{6.20}$$

where B_h depends on the number of elements from the space discretization. Hence $\mathbf{E}\|W(t)\|^p$ is finite, independent of the choice of γ_m. When $D \subset \mathbb{R}^1$, the noise $W(t)$ is a real-valued Brownian motion.

We implement the numerical experiments with the same conditions and setups in **Chapter 4** section 4.4. The proxy of the "true" solution is computed with the time step $k = 2^{-10}$ and the space step $h = 2^{-8}$. In computation we choose the time step k by the *Courant-Friedrichs-Lewy (CFL) condition* to achieve stability of the FEM scheme. For example, the *CFL* condition says that the time step k should be chosen such that

$$c_e \frac{k}{h} < 1,$$

where c_e is the speed of heat conduction over the material (refer [4] for details). Hence, the choice of time step and space step follows once one is fixed.

Similar to the numerical experiments in **Chapter 4** section 4.4, we expect that the average of "true" solutions from the simulations converges to the mean $\mathbf{E}(u)$, where $\mathbf{E}(u)$ is the solution of the deterministic counterpart of (6.18) with homogeneous Dirichlet boundary conditions. The average of 100 simulations of u and the expected value $\mathbf{E}(u)$ is shown in Figure 6.1.

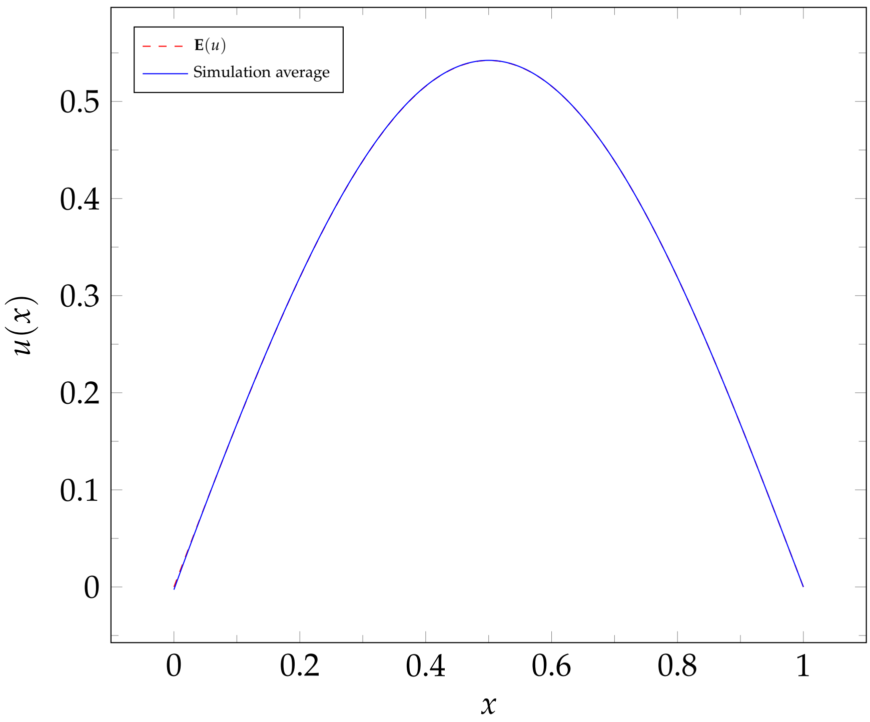


Figure 6.1: Average of simulations and $\mathbf{E}(u)$

Only minor divergence between the simulation average and the theoretical mean can be observed at the boundary point 0. Within the computational domain $(0, 1)$, the simulation average and the expectation $\mathbf{E}(u)$ are well overlapped. It coincides with the conclusions in [1], [39] and [7]. Even though the boundary data is highly irregular, the solution within the domain is smooth.

This result can be seen even more obviously in Figure 6.2. In the plot, we show 100 simulations of the solution u as well as the average and 2 stand deviation of the simulated data. The solution only behaves ”badly” at the singular point 0.

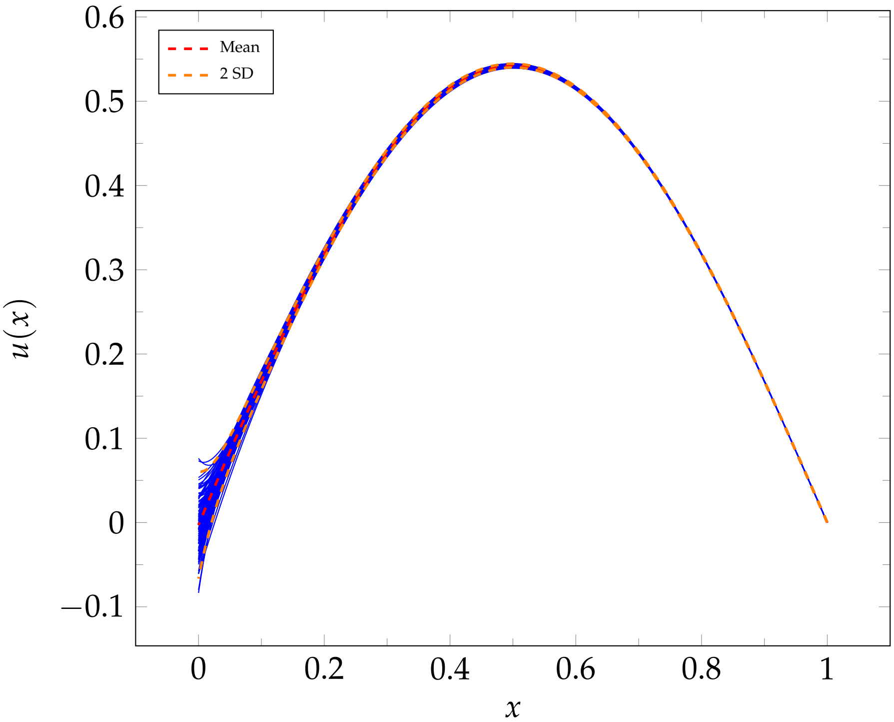
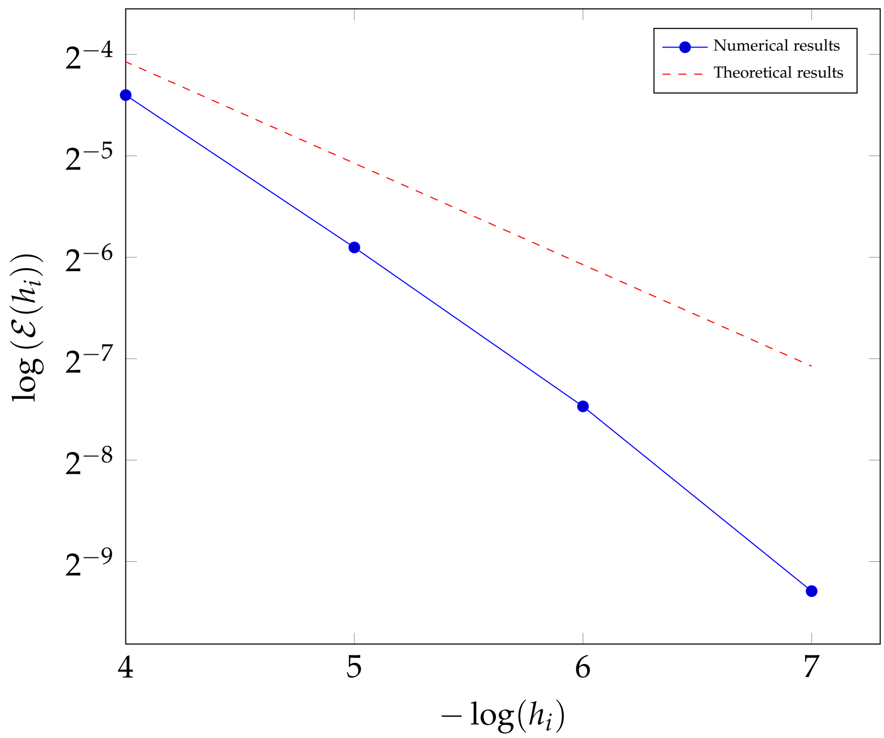


Figure 6.2: Distribution of simulations of solution

Now we verify the convergence rate in space shown in Theorem 6.5 and 6.6 by computing the L^2-norm of the errors with respect to the different spatial mesh sizes. The results are shown in Figure 6.3a and Table 6.3b.

(a) L^2-norm errors in different mesh size



(b) Ratios of errors with different mesh size

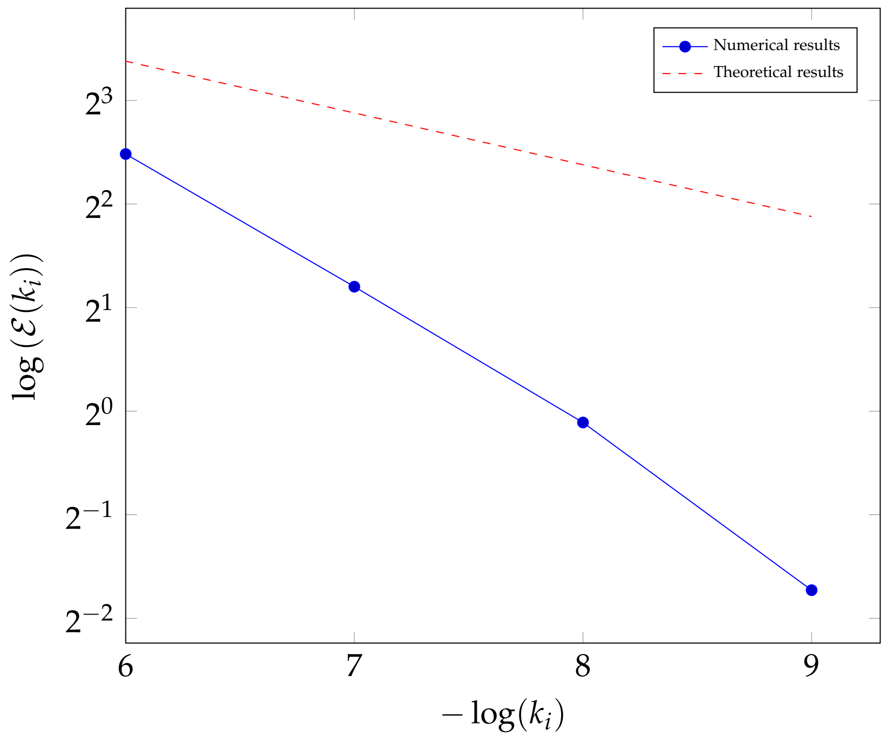
i	h_i	$\mathcal{E}(h_i)$	$\mathcal{E}(h_i)/\mathcal{E}(h_{i+1})$
4	2^{-4}	$4.7343 \cdot 10^{-2}$	2.83
5	2^{-5}	$1.6721 \cdot 10^{-2}$	2.96
6	2^{-6}	$5.6450 \cdot 10^{-3}$	3.53
7	2^{-7}	$1.5976 \cdot 10^{-3}$	

Figure 6.3: Errors in L^2-norm with time stepping size $k = 2^{-10}$

Under the same assumptions, we showed in the Neumann boundary noise case in **Chapter 4**, that the solution $u \in H^1(D)$ for each simulation has the convergence rate in space of h^1 rather than h^r. We also know that the solution within the domain $u \in C^\infty(D_\epsilon)$ where $\epsilon > 0$ denotes the distance away from the boundary (see [7]). The results in Figure 6.3a and Table 6.3b demonstrate exactly this point.

Similarly, the L^2-norm of errors with respect to the different time stepping sizes are shown in Figure 6.4a and Table 6.4b. The theoretical convergence rate in time, given by Theorem 6.6, is $k^{\frac{r}{2}}$. However, we expect it to be $k^{\frac{1}{2}}$ due to the setup of our test problem, as the actual problem in simulations is more regular in both space and time than required.

(a) L^2-norm errors in different time step



(b) Ratios of errors with different time step

i	k_i	$\mathcal{E}(k_i)$	$\mathcal{E}(k_i)/\mathcal{E}(k_{i+1})$
6	2^{-6}	5.5890	2.43
7	2^{-7}	2.3002	2.50
8	2^{-8}	0.9270	3.08
9	2^{-9}	0.3019	

Figure 6.4: Errors in L^2-norm with mesh size $h = 2^{-8}$

This can be seen in the tables and plots above where the convergence rates in the numerical experiment are better than the theoretical ones. It is due to the fact that the regularity of numerical solution is data dependent. With the smooth initial condition and the necessary truncation of white noise (6.20) in the simulation, it is reasonable to expect a more regular solution and consequently better convergence rates.

Finally we present the computational results of $\mathbf{E}(u)$ and $\mathbf{E}(u^2)$ at the point $t = 1, x = 0.5$ with different space mesh size and time stepping size in Table 6.1. The solution converges as $h, k \to 0$, which coincides with the numerical analysis in the previous sections of the chapter.

h	k	$\mathbf{E}(u(1,0.5))$	$\mathbf{E}(u(1,0.5)^2)$
1/4	1/4	$5.9755 \cdot 10^1$	$3.5708 \cdot 10^3$
1/4	1/8	$1.2847 \cdot 10^1$	$1.6506 \cdot 10^2$
1/4	1/16	3.3437	$1.1183 \cdot 10^1$
1/4	1/32	1.2430	1.5460
1/8	1/8	$1.5248 \cdot 10^1$	$2.3251 \cdot 10^2$
1/8	1/16	4.2412	$1.7990 \cdot 10^1$
1/8	1/32	1.6691	2.7869
1/16	1/16	4.4989	$2.0242 \cdot 10^1$
1/16	1/32	1.7953	3.2241
1/256	1/1024	0.5432	0.2941

Table 6.1: $\mathbf{E}(u)$ and $\mathbf{E}(u^2)$ at $t = 1, x = 0.5$

Chapter 7

Conclusions and extensions

In the thesis, we have applied numerical approximation methods to solving SPDEs with boundary noise. For the semi-linear SPDEs with Neumann boundary noise, a backward time stepping with Galerkin FEM is considered, and a numerical scheme with discontinuous Galerkin time stepping is developed for solving the one-dimensional heat equation with the Dirichlet boundary noise. Further, we have shown:

1. Optimal convergence rates in approximating the solutions to the semi-linear SPDEs with Neumann boundary noise;

2. The usage of SPDEs in modeling solute dynamics in arteries;

3. Optimal convergence rates in approximating the solutions to the heat equation with the white noise Dirichlet boundary condition in the one dimensional case.

We have also verified our theoretical results through numerical experiments. The results from our numerical experiments are consistent with the rates determined through our theoretical analysis. Moreover, it has been shown that by truncating infinite series representation of the noise, the convergence rates are much better than the theoretical ones due to the "smoothing" effect. As such, we have gained insight in understanding the barriers and bottle necks of numerical schemes for boundary noise problems by investigating the problems both theoretically and numerically. We have the following conclusions from the proofs in the previous chapters:

1. There exists an inverse relationship between time stepping size k and time span T so that the temporal convergence rate is constrained by this relationship;

2. There is a trade-off on the regularities between time and space. However, it is not without limit;
3. For the equations with Dirichlet boundary noise, even though increasing the power of weight can increase the convergence rate, it has a limit.

Therefore, in this thesis, we have not only shown the convergence rates but also answered the following questions:

1. What is the relationship between regularity and convergence?

   Throughout the thesis, we present the necessary assumptions and lemmas in clarifying the relationship between the convergence rates and the regularity of solutions w.r.t. both space and time. In the proofs of Theorem 4.6 and Theorem 4.9, we show how the uniform Lipschitz continuity of all the terms 3.4 contribute to the convergence rates. Especially the Lipschitz continuity in time of order $\frac{r}{2}$ in Assumption 3.2 - 3.4 is essential in achieving the optimal convergence rate not only in time but in space.

2. Why is the convergence rate as it is?

   In the proofs of Theorem 6.4 - 6.6, we explain that the the convergence rates depend on the weight of the weighed Sobolev space in addition to the regularity of the solution.

   Moreover, we present results from numerical experiments and explain how the convergence rates depend consistently on the data. We observe that the numerical results are generally better than the theoretical ones and explain that this happens due to the data dependency in the simulations.

3. What are the conditions for obtaining the optimal convergence rate?

   In **Chapter 4**, we note that for the semilinear SPDEs with Neumann boundary noise, besides the uniformly Lipschitz continuity of all the terms in 3.4, Assumption 4.5 is essential to achieve the optimal convergence rate. It remarks the minimum regularity requirement of the boundary noise, i.e., $(\delta + A)N_\delta \bar{c}(t, x, U(t, x))\frac{\partial w_2}{\partial t}(t, x) \in H^{\alpha-2}(D)$ where $\alpha > 1 + r$ when $U(t, x) \in \dot{H}^r$ with $r < 1$.

In **Chapter 5**, we present the conditions under which the solute dynamic within the arteries is modeled by SPDEs with the Neumann boundary noise. Hence this more difficult applied problem can be solved in the framework proposed in **Chapter 4**.

In **Chapter 6**, we prove that the regularity of the solution is of the order $0 < r < 1$ in Theorem 6.4 such that it is sensible to discuss the convergence rates in the numerical approximation. We also introduce a Galerkin time stepping scheme to obtain a convergence rate in time of $\frac{r}{2}$.

4. What are the constraints or limits on convergence?

   We notice that the trade-off between the regularity in time and that in space is the center of all the lemmas and theorems. It is implied in Lemma 4.3 and Lemma 4.7, e.g., if the approximate solution $F_h(t)x$ has a less regular x in space, it requires more regularity in time since the estimate of $\|F_h(t)x\|$ blows up in the order $\frac{\nu}{2}$ when $t \to 0$. It leads back to the assumptions of Lipschitz continuity of order $\frac{r}{2}$ in time. The fact is also shown in the estimates of heat kernel p_D (6.1 and 6.2). However, this leads to constrains in both lemmas as $\nu \leq 1$.

   Limits are also presented in the regularity of the solutions. In the heat equation with Dirichlet boundary noise, the regularity of solution is constrained by the weight order $r < \frac{\varrho}{4}$. However we can only increase the regularity of the solution to the limit $\varrho < 1$. We note in the thesis that this constraint on ϱ is required in the estimate. This is also shown explicitly by (6.16) in the proof of Theorem 6.5. The integral in (6.16) would explode if we did not have $\varrho < 1$, which would result in convergence rate in space exploding.

We note that these questions could not have been answered by simulation alone. For example, better convergence rates are achieved by the simulation in [22]. Burgers equations discussed in [22] fall into the category of semi-linear PDEs with Neumann boundary conditions. In that paper, a convergence rate of $\frac{1}{2}$ in time is achieved consistently over all the simulation examples. Hence, the authors conclude that the convergence rate of Burgers equations in time is $\frac{1}{2}$. We show in the thesis through theoretical analysis that for the semi-linear PDEs with Neumann boundary conditions, the convergence rate in time is

$\frac{r}{2}$ with $r < 1$. We also show experimental results better than the theoretical ones in the thesis due to the difficulty in simulating true white noise on the boundary in low dimensional examples. We acknowledge this discrepancy and explain why this occurs.

7.1 Future Work

Since we have understood and identified the limits of the numerical schemes for SPDEs with boundary noise, our work can be extended in the following aspects:

1. Possibility of extending the numerical scheme to higher dimensional equations with Dirichlet white noise boundary conditions. The issue with the higher dimensional case is that the increasing of the dimension will add more irregularity through time. As it is shown in the estimate of derivative of the heat kernel in Lemma 5.9 [39], for the multi-dimensional case,

$$\left|\frac{\partial^{m+k} p_D}{\partial y^k \partial x^m}(s,t,y,x)\right| \leq C(t-s)^{-\frac{m+k+d}{2}} \exp\left(-\frac{|y-x|^2}{c(t-s)}\right),$$

   where d denotes the dimension in space. The proof of Theorem 6.4 shows that the regularity of the solution is constrained by the above factor when the term II is essentially bounded by the integral

$$\int_0^t (t-s)^{-d-2r+\alpha} ds,$$

   with $r < 1$ and $\alpha > 2r$. When $d > 1$, the integral w.r.t time will blow up to ∞. Also the rate of blowing-up close to the boundary is not well understood. If there exists a special numerical scheme that could solve this issue it would be worth looking at.

2. Consider other forms of weight function, e.g., exponential functions. The convergence should happen "faster" if the measure on the boundary goes to zero faster. Since we have known there is a limit with the power of weight function, an exponential function could be a choice to

avoid some problems, but there might be undesired trade-offs.

3. Other time stepping techniques for tackling the bottle-neck of convergence rate in time [27]. We have known that the barrier of convergence rate in time lies at $\frac{r}{2}$. It could be very close to $\frac{1}{2}$ but will never achieve this. To refine the time stepping size to achieve more accurate results is very time consuming and computationally expensive. Hence another time stepping technique could possibly be help in breaking this barrier.

4. Can we possibly find a scheme that handles the trade-off between space and time? Again we cannot force regularity of the noise without limit. We cannot trade the regularity in space with that in time in an unlimited way either. These are hard constraints on the convergence rate. Can we break free from the trade-off in some way?

Appendix A

Lemma A.1. *For $t_j = jk, j = 1, \ldots, N_k$ such that $N_k k \leq T < (N_k + 1)k$ where $T > 0$ and $k > 0$. Let $C_1, C_2 \geq 0$ and $\{\phi_j\}_{j=1,\ldots,N_k}$ be a nonnegative sequence.*

If for $\beta \in (0, 1]$ we have

$$\phi_j \leq C_1 + C_2 \sum_{i=0}^{j-1} t_{j-i}^{-1+\beta} \phi_i \qquad \forall j = 1, \ldots, N_k,$$

then there exists a constant $C = C(C_2, T, \beta)$ not depending on k such that

$$\phi_j \leq C C_1 \qquad \forall j = 1, \ldots, N_k.$$

Lemma A.2 (Gronwall's Lemma)**.** *[30, Lemma A.2] Let $T > 0$ and $C_1, C_2 \geq 0$ and let $\phi : [0, T] \to \mathbb{R}$ be a nonnegative and continuous function. Let $\beta > 0$. If we have*

$$\phi(t) \leq C_1 + C_2 \int_0^t (t - s)^{-1+\beta} \phi(s) ds \quad \forall t \in (0, T],$$

then there exists a constant $C = C(C_2, T, \beta)$ such that

$$\phi(t) \leq C C_1, \quad \forall t \in (0, T].$$

Lemma A.3 (Intermediate case of **Lemma 4.3 (ii)**)**.**

Proof. For $0 < \mu < 1$, we denote $u = E(t)x$ and expand u in the form

$$u = \sum_{h^2 \lambda_m \leq 1} (u, \varphi_m) \varphi_m + \sum_{h^2 \lambda_m > 1} (u, \varphi_m) \varphi_m =: u_I + u_{II},$$

where λ_m and φ_m are the eigenvalues and eigenvectors of the operator A.

Then for the term u_I , with Assumption 4.1 and Proposition 4.2, we have

$$\begin{aligned} \|(P_h - \mathrm{Id})u_I\|^2 &\le Ch^2\|u\|_1^2 = Ch^2 \sum_{h^2\lambda_m \le 1} \lambda_m (u, \varphi_m)^2 \\ &\le Ch^{2(1-\epsilon)} \sum_{m=1}^{\infty} \lambda_m^{1-\epsilon}(u, \varphi_m)^2 = Ch^{2\mu}\|u\|_\mu^2, \end{aligned} \tag{A.1}$$

where $0 < \epsilon < 1$ and $\mu = 1 - \epsilon$. Similarly for the term u_{II}, the estimate

$$\begin{aligned} \|(P_h - \mathrm{Id})u_{II}\|^2 &\le C\|u\|^2 = C \sum_{h^2\lambda_m > 1} (u, \varphi_m)^2 \\ &\le Ch^{2\mu} \sum_{m=1}^{\infty} \lambda^{\mu}(u, \varphi_m)^2 = Ch^{2\mu}\|u\|_\mu^2. \end{aligned} \tag{A.2}$$

Together (A.1) and (A.2) complete the proof. □

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